$|\alpha\rangle \rightarrow a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$|\beta\rangle = \hat{T}|\alpha\rangle$

$b = Ta = \begin{pmatrix} t_{11} & t_{12} & & \\ t_{21} & t_{22} & & \\ & & \ddots & t_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_n \end{pmatrix}$

$\int |\psi|^2 dx = 1$

Square-integrable

$\int_a^b |f(x)|^2 dx < \infty$

$L^2(a,b)$ Hilbert space

Wave functions live in Hilbert space

inner product of two functions

$\langle f|g\rangle \equiv \int_a^b f(x)^* g(x)\, dx$

$\left| \int_a^b f(x)^* g(x)\, dx \right| \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx}$

$\langle g|f\rangle = \langle f|g\rangle^*$

$\langle f|f\rangle = \int_a^b |f(x)|^2 dx$

$\langle f_m|f_n\rangle = \delta_{mn}$

$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$

Complete inner product space

↓

Hilbert space.

$\langle Q \rangle = \int \psi^* \hat{Q} \psi \, dx$

$= \langle \psi | \hat{Q} \psi \rangle$

$\langle Q \rangle = \langle Q \rangle^*$

$\langle \psi | \hat{Q} \psi \rangle = \langle \hat{Q} \psi | \psi \rangle$

$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$.

$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$

$\langle f | \hat{p} g \rangle = \int_{-\infty}^{\infty} f^* (-i\hbar) \frac{dg}{dx} dx$

$= (-i\hbar) f^* g \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (-i\hbar \frac{df}{dx})^* g \, dx$

$= 0 + \int_{-\infty}^{\infty} (-i\hbar \frac{df}{dx})^* g \, dx$

$= \langle \hat{p} f | g \rangle$

$\langle f | \hat{p} g \rangle = \langle \hat{p} f | g \rangle$

$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$

$\langle f | \hat{Q} g \rangle = \langle \hat{Q}^\dagger f | g \rangle$

$\hat{Q} = \hat{Q}^\dagger$

$$\sigma^2 = \langle (Q-\langle Q\rangle)^2 \rangle$$

$$= \langle \psi | (\hat{Q}-q)^2 \psi \rangle$$

$$= \langle (\hat{Q}-q)\psi | (\hat{Q}-q)\psi \rangle$$

$$= 0$$

$$\langle Q \rangle = q$$

$$\hat{Q}\psi = q\psi$$

Determinate states of Q are eigenfunctions of $\hat{Q}$

$$A\psi = E\psi$$

$$\hat{Q} = i\frac{d}{dq}$$

$$\langle f | \hat{Q} g \rangle = \int f^* i \frac{dg}{dq} dq$$

$$= i f^* g \Big|_0^{2\pi}$$

$$i\frac{d}{dq} f(q) = q f(q)$$

$$f(q) = Ae^{-iqq}$$

$$i\frac{d}{dq}(Ae^{-iqq}) = q(Ae^{-iqq})$$

$$i(-iq)Ae^{-iqq} = q(Ae^{-iqq})$$

$$qAe^{-iqq} = qAe^{-iqq}$$

✓

$\hat{Q}f = qf$

$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$

$q\langle f|f\rangle = q^*\langle f|f\rangle$

$q = q^*$

q is real

$\hat{Q}f = qf \quad \hat{Q}g = q'g$

$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$

$q'\langle f|g\rangle = q^*\langle f|g\rangle$

$q' \neq q$

$\Rightarrow \langle f|g\rangle = 0$

$-i\hbar \frac{d}{dx} f_p(x) = p f_p(x)$

$-i\hbar r = p$

$r = -\frac{p}{i\hbar}$

$r = -\frac{i}{i^2\hbar} p$

$r = -\frac{ip}{\hbar}$

$f_p = Ae^{-ipx/\hbar} + Be^{ipx/\hbar}$

$f_p = Ae^{ipx/\hbar}$

$$\int_{-\infty}^{\infty} f^*_{p'}(x) f_p(x)\,dx$$

$$= |A|^2 \int e^{i(p-p')x/\hbar}$$

$$= |A|^2 \, 2\pi\hbar\, \delta(p-p')$$

$$\langle f_{p'} | f_p \rangle = \delta(p-p')$$

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x)\,dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar}\,dp$$

$$\langle f_{p'} | f \rangle = \int_{-\infty}^{\infty} c(p) \langle f_{p'} | f_p \rangle\,dp$$

$$= \int_{-\infty}^{\infty} c(p) \delta(p-p')\,dp$$

$$= c(p')$$

$$\lambda = \frac{2\pi\hbar}{p}$$

$$\hat{x}\, g_y(x) = x\, g_y(x) = y\, g_y(x)$$

$g_y(x)$ the eigenfunction
y the eigenvalue

$$g_y(x) = A\,\delta(x-y)$$

$$\int_{-\infty}^{\infty} g^*_{y'}(x)\, g_y(x)\,dx$$

$$= |A|^2 \int_{-\infty}^{\infty} \delta(x-y')\,\delta(x-y)\,dx$$

$$= |A|^2\, \delta(y-y')$$

$A=1$

$g_y(x) = \delta(x-y)$

$\langle g_{y'} | g_y \rangle = \delta(y-y')$

$f(x) = \int_{-\infty}^{\infty} c(y)\, g_y(x)\, dy$

$= \int_{-\infty}^{\infty} c(y)\, \delta(x-y)\, dy$

$c(y) = f(y)$

Generalized Statistical Interpretation

$Q(x,p)$ $\quad \psi(x,t)$

$\hat{Q}(x, -i\hbar \, d/dx)$

Spectrum of $\hat{Q}$ $\Rightarrow$ discrete

eigenvalue q_n

eigenfunction $f_n(x)$

$|C_n|^2 \quad C_n = \langle f_n | \psi \rangle$

$|C(z)|\, dz \quad C(z) = \langle f_z | \psi \rangle$

$\psi(x,t) = \sum_n C_n(t)\, f_n(x)$

$C_n(t) = \langle f_n | \psi \rangle = \int f_n(x)^* \, \psi(x,t)\, dx$

$\sum_n |C_n|^2 = 1$

$$1 = \langle \psi | \psi \rangle = \left\langle \left(\sum_{n'} c_{n'} f_{n'} \right) \middle| \left(\sum_n c_n f_n \right) \right\rangle$$

$$= \sum_{n'} \sum_n c_{n'}^* c_n \langle f_{n'} | f_n \rangle$$

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

$$\langle Q \rangle = \langle \psi | \hat{Q} \psi \rangle = \left\langle \left(\sum_{n'} c_{n'} f_{n'} \right) \middle| \left(\hat{Q} \sum_n c_n f_n \right) \right\rangle$$

$$\hat{Q} f_n = q f_n$$

$$\langle Q \rangle = \sum_{n'} \sum_n c_{n'}^* c_n q_n \langle f_{n'} | f_n \rangle$$

$$= \sum_{n'} \sum_n c_{n'}^* c_n q_n \delta_{n'n}$$

$$= \sum_n q_n |c_n|^2$$

$$q_y(x) = \delta(x - y)$$

$$c(y) = \langle q_y | \psi \rangle$$

$$= \int_{-\infty}^{\infty} \delta(x - y) \psi(x, t) \, dx$$

$$= \psi(y, t)$$

Probability get result in range dy

$$|\psi(y,t)|^2 dy$$

(Dirac-orthonormalized) eigenfunctions of the momentum operator

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x,t)\, dt$$

momentum space wave function

$\phi(p,t)$

Fourier transform of the position space wave function $\psi(x,t)$

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x,t)\, dx$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \phi(p,t)\, dp$$

$|\phi(p,t)|^2\, dp$

Uncertainty Principle

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle)\psi \,|\, (\hat{A} - \langle A \rangle)\psi \rangle$$

$$= \langle f | f \rangle$$

$$f = (\hat{A} - \langle A \rangle)\psi$$

$$\sigma_B^2 = \langle g | g \rangle$$

$$g = (\hat{B} - \langle B \rangle)\psi$$

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

$$z = \langle f|g\rangle$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2$$

$$(\hat{A} - \langle A\rangle)$$

$$\langle f|g\rangle = \langle (\hat{A} - \langle A\rangle)\psi | (\hat{B} - \langle B\rangle)\psi\rangle$$

$$= \langle \psi | (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)|\psi\rangle$$

$$= \langle \psi | (\hat{A}\hat{B} - \langle A\rangle\hat{B} - \hat{A}\langle B\rangle + \langle A\rangle\langle B\rangle)|\psi\rangle$$

$$= \langle \psi|\hat{A}\hat{B}\psi\rangle - \langle A\rangle\langle\psi|B\psi\rangle - \langle B\rangle\langle\psi|\hat{A}\psi\rangle + \langle A\rangle\langle B\rangle\langle\psi|\psi\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle - \langle B\rangle\langle A\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle B\rangle\langle A\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle$$

$$\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle$$

$$\langle g|f\rangle = \langle (\hat{B} - \langle B\rangle)\psi | (\hat{A} - \langle A\rangle)\psi\rangle$$

$$= \langle \psi | (\hat{B} - \langle B\rangle)(\hat{A} - \langle A\rangle)\psi\rangle$$

$$= \langle \psi | (\hat{B}\hat{A} - \hat{B}\langle A\rangle - \langle B\rangle\hat{A} + \langle B\rangle\langle A\rangle)\psi\rangle$$

$$= \langle \psi|\hat{B}\hat{A}\psi\rangle - \langle A\rangle\langle\psi|B\psi\rangle - \langle B\rangle\langle\psi|\hat{A}\psi\rangle$$
$$\langle\psi|\langle B\rangle\langle A\rangle\psi\rangle$$

$= \hat{B}\hat{A} - \langle A\rangle\langle B\rangle - \langle B\rangle\langle A\rangle + \langle BA\rangle$

$= \hat{B}\hat{A} - \langle A\rangle\langle B\rangle$

$\langle f|g\rangle - \langle g|f\rangle = \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle - \langle \hat{B}\hat{A}\rangle + \langle A\rangle\langle B\rangle$

$= \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle$

$= \langle [\hat{A}, \hat{B}]\rangle$

$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

$\overline{\Delta A}^2 \overline{\Delta B}^2 \geq \left| \frac{1}{2i} [\langle f|g\rangle - \langle g|f\rangle] \right|^2$

$= \left| \frac{1}{2i} \langle [\hat{A}, \hat{B}]\rangle \right|^2$

uncertainty principle.

$\hat{A} = x$

$\hat{B} = -i\hbar \, d/dx$

$[\hat{x}, \hat{p}] = -i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i}\,i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

There is an "uncertainty principle" for every pair of observables whose operators do not commute – we called them incompatible observables.

They cannot have a complete set of common eigenfunctions.

Compatible (commuting) observables do admit complete sets of simultaneous eigenfunctions.

The minimum-uncertainty Wave Packet

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

The Schwarz inequality
→ equality

↓

When one function is a multiple of another

$g(x) = cf(x)$

$\mathrm{Re}(z) = 0$

$\mathrm{Re}\langle f|g\rangle = \mathrm{Re}(c\langle f|f\rangle) = 0$

$g(x) = iaf(x) \quad a \in \mathbb{R}$

$\left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\psi = ia(x - \langle x\rangle)\psi$

$\psi(x) = Ae^{-a(x-\langle x\rangle)^2/2\hbar}\, e^{i\langle p\rangle x/\hbar}$

The Energy-Time Uncertainty Principle

position-momentum uncertainty principle.

$\Delta x \Delta p \geq \frac{\hbar}{2}$

energy-time uncertainty principle

$\Delta t \Delta E \geq \frac{\hbar}{2}$

$Q(x, p, t)$

$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle\psi|\hat{Q}\psi\rangle$

$= \langle\frac{\partial\psi}{\partial t}|\hat{Q}\psi\rangle + \langle\psi|\frac{\partial\hat{Q}}{\partial t}\psi\rangle + \langle\psi|\hat{Q}\frac{\partial\psi}{\partial t}\rangle$

$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \quad \frac{\partial\psi}{\partial t} = \frac{\hat{H}}{i\hbar}\psi$

$H = P^2/2m + V$

$\frac{\partial \hat{Q}}{\partial t}\psi = \hat{Q}\psi - \hat{Q}\frac{\partial \psi}{\partial t}$

$\frac{d}{dt}\langle Q\rangle = \langle \frac{\hat{H}}{i\hbar}\psi | \hat{Q}\psi\rangle + \langle \psi | \frac{\partial \hat{Q}}{\partial t}\psi\rangle + \langle \psi | \hat{Q}\frac{\hat{H}}{i\hbar}\psi\rangle$

$= -\frac{1}{i\hbar}\langle \hat{H}\psi | \hat{Q}\psi\rangle + \frac{1}{i\hbar}\langle \psi | \hat{Q}\hat{H}\psi\rangle + \langle \frac{\partial \hat{Q}}{\partial t}\rangle$

$\langle \hat{H}\psi | \hat{Q}\psi\rangle = \langle \psi | \hat{Q}\hat{H}\psi\rangle$

$\frac{d}{dt}\langle Q\rangle = \frac{1}{i\hbar}\langle [\hat{H}, \hat{Q}]\rangle + \langle \frac{\partial \hat{Q}}{\partial t}\rangle$

generalized Ehrenfest theorem

$A = H \quad B = Q$

$\sigma_H^2 \sigma_Q^2 \geq \left| \frac{1}{2i}\langle [\hat{H}, \hat{Q}]\rangle \right|^2$

$= \left| \frac{1}{2i}\frac{\hbar}{i}\frac{d\langle Q\rangle}{dt} \right|^2$

$= \left(\frac{\hbar}{2}\right)^2 \left(\frac{d\langle Q\rangle}{dt}\right)^2$

$\sigma_H \sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$

$\sigma_E = \sigma_H$

$E \sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$

$E \frac{\sigma_Q}{\left|\frac{d\langle Q\rangle}{dt}\right|} \geq \frac{\hbar}{2}$

$E \Delta t \geq \frac{\hbar}{2}$

$$\Delta t = \frac{\delta q}{\left|\frac{dq}{dt}\right|}$$

$$\delta q = \left|\frac{dq}{dt}\right| \Delta t$$

$A_x = \hat{\imath} \cdot \vec{A}$ $\quad A_{x'} = \hat{\imath}' \cdot \vec{A}$

$A_y = \hat{\jmath} \cdot \vec{A}$ $\quad A_{y'} = \hat{\jmath}' \cdot \vec{A}$

$\{\hat{\imath}, \hat{\jmath}\}$ $\quad \{\hat{\imath}', \hat{\jmath}'\}$

$|S(t)\rangle$

$|S(t)\rangle$

$\psi(x,t) = \langle x | S(t) \rangle$

$\psi(x,t)$ is the x component in the expansion of $|S(t)\rangle$ in the basis of position eigenfunctions

$\phi(p,t) = \langle p | S(t) \rangle$

$c_n(t) = \langle n | S(t) \rangle$

$$|S(t)\rangle \rightarrow \int \psi(y,t)\, \delta(x-y)\, dy$$

$$= \int \phi(p,t) \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\, dp$$

$$= \sum c_n e^{-iE_n t/\hbar} \psi_n(x)$$

$$|\beta\rangle = \hat{Q}|\alpha\rangle$$

Operators (representing observables) are linear transformations on the Hilbert space - they "transform" one vector into another:

$$|x\rangle = \sum_n a_n |e_n\rangle$$

$$|\beta\rangle = \sum_n b_n |e_n\rangle$$

$$a_n = \langle e_n | \alpha \rangle$$

$$b_n = \langle e_n | \beta \rangle$$

$$\langle e_m | \hat{Q} | e_n \rangle \equiv Q_{mn}$$

$$\sum_n b_n |e_n\rangle = \sum_n a_n \hat{Q} |e_n\rangle$$

$$\sum_n b_n \langle e_m | e_n \rangle = \sum_n a_n \langle e_m | \hat{Q} | e_n \rangle$$

$$\langle e_m | e_n \rangle = \delta_{mn}$$

$$b_m = \sum_n Q_{mn} a_n$$

Ex 3.8

$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$|S\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix}a\\b\end{pmatrix}$

$|a|^2 + |b|^2 = 1$

$H = \begin{pmatrix}h & g\\ g & h\end{pmatrix}$

$i\hbar \frac{d}{dt}|S(t)\rangle = \hat{H}|S(t)\rangle$

$\hat{H}|S\rangle = E|S\rangle$

$\det\begin{pmatrix}h-E & g\\ g & h-E\end{pmatrix} = (h-E)^2 - g^2 = 0$

$(h-E)^2 = g^2$

$h - E = \mp g$

$E_\pm = h \pm g$

$\begin{pmatrix}h & g\\ g & h\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (h\pm g)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow h\alpha + g\beta = (h\pm g)\alpha$

$\beta = \pm\alpha$

$|S_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$

$|S(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|S_+\rangle + |S_-\rangle)$

$$|S(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i(\pi+q)t/\pi}|S_+\rangle + e^{-i(\pi-q)t/\pi}|S_-\rangle\right]$$

$$= \frac{1}{2}e^{-i\pi t/\pi}\left[e^{-iqt/\pi}\begin{pmatrix}1\\1\end{pmatrix} + e^{iqt/\pi}\begin{pmatrix}-1\\1\end{pmatrix}\right]$$

$$= \frac{1}{2}e^{-i\pi t/\pi}\begin{pmatrix} e^{-iqt/\pi}+e^{iqt/\pi} \\ e^{-iqt/\pi}-e^{iqt/\pi}\end{pmatrix}$$

$$= e^{-i\pi t/\pi}\begin{pmatrix}\cos(qt/\pi)\\ -i\sin(qt/\pi)\end{pmatrix}$$

$\hat{x}$ (the position operator) $\rightarrow \begin{cases} x & \text{(in position space)} \\ i\hbar\,\partial/\partial p & \text{(in momentum space)}\end{cases}$

$\hat{p}$ (the momentum operator) $\rightarrow \begin{cases} -i\hbar\,\partial/\partial x & \text{(in position space)} \\ p & \text{(in momentum space)}\end{cases}$

Dirac Notation

bra $\langle\alpha|$ ← linear function of vectors

ket $|\beta\rangle$ ← vector

When an operator hits a vector
it delivers another vector

When a bra hits a vector,
it delivers a number.

$$\langle f | = \int f^* [\dots] dx$$

$$|a\rangle \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\langle B | \rightarrow (b_1^* \; b_2^* \dots b_n^*)$$

$$\langle B | a \rangle = b_1^* a_1 + b_2^* a_2 + \dots + b_n^* a_n$$

The collection of all bras constitutes another vector space called dual space.

$$\hat{P} \equiv |\alpha\rangle\langle\alpha|$$

$$\hat{P}|\beta\rangle = (\langle\alpha|\beta\rangle)|\alpha\rangle$$

$$\langle e_m | e_n \rangle = \delta_{mn}$$

$$\sum_n |e_n\rangle\langle e_n| = 1$$

$$\sum_n (\langle e_n|\alpha\rangle)|e_n\rangle = |\alpha\rangle$$

$$\langle e_z | e_{z'} \rangle = \delta(z - z')$$

$$\int |e_z\rangle\langle e_z| \, dz = 1$$

Completeness

$$\sum_n |e_n\rangle\langle e_n| = 1$$

$$\int |e_z\rangle\langle e_z| \, dz = 1$$

$$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$$

$$\langle f|\hat{Q}|f\rangle$$

$$\langle \hat{Q}f|$$

$$\hat{Q}|\alpha\rangle = |\beta\rangle$$

$$(\hat{Q}+\hat{R})|\alpha\rangle = \hat{Q}|\alpha\rangle + \hat{R}|\alpha\rangle$$

$$\hat{Q}\hat{R}|\alpha\rangle = \hat{Q}(\hat{R}|\alpha\rangle)$$

$$e^{\hat{Q}} \equiv 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \cdots$$

$$\frac{1}{1-\hat{Q}} \equiv 1 + \hat{Q} + \hat{Q}^2 + \hat{Q}^3 + \hat{Q}^4 + \cdots$$

$$\ln(1+\hat{Q}) \equiv \hat{Q} - \frac{1}{2}\hat{Q}^2 + \frac{1}{3}\hat{Q}^3 - \frac{1}{4}\hat{Q}^4 + \cdots$$

Changing Bases in Dirac Notation

position eigenstates $|x\rangle$

momentum eigenstates $|p\rangle$

energy eigenstates $|n\rangle$.

(Assume discrete)

$$\mathbb{1} = \int dx\, |x\rangle\langle x|$$

$$\mathbb{1} = \int dp\, |p\rangle\langle p|$$

$$\mathbb{1} = \sum |n\rangle\langle n|$$

$$|S(t)\rangle = \int dx\, |x\rangle\langle x|S(t)\rangle = \int dx\, \psi(x,t)\,|x\rangle dx$$

$$|S(t)\rangle = \int dp\, |p\rangle\langle p|S(t)\rangle = \int dp\, \phi(p,t)|p\rangle dp$$

$$|S(t)\rangle = \sum_n |n\rangle\langle n|S(t)\rangle \equiv \sum c_n(t)\,|n\rangle$$

Ex 3.9

$$\phi(p,t) = \langle p|S(t)\rangle$$

$$\psi(x,t) = \langle x|S(t)\rangle$$

$$\phi(p,t) = |p|S(t)\rangle$$

$$= \langle p|\left(\int dx\, |x\rangle\langle x|\right)|S(t)\rangle$$

$$= \int dx\, \langle p|x\rangle\langle x|S(t)\rangle$$

$$= \int dx\, \langle p|x\rangle\, \psi(x,t)\, dx$$

$$\langle p|x\rangle = \langle x|p\rangle^* = [f_p(x)]^*$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

$$\phi(p,t) = \int \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\, \psi(x,t)\, dx$$

$$\hat{x} \to x$$

$$\hat{p} \to i\hbar \frac{\partial}{\partial p}$$

$\langle x|\hat{x}|\psi(t)\rangle$ = action of position operator in x basis

$$= x\,\psi(x,t)$$

$\langle p|\hat{x}|\psi(t)\rangle$ = action of position operator in p basis $= i\hbar \frac{\partial \phi}{\partial p}$