Multipole Expansion Review

$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\vec p\cdot\hat r}{r^2} + \frac{Q_{ij}\hat r_i \hat r_j}{r^3} + \cdots\right)$$

monopole, dipole, quadrupole

$$\boxed{V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{\vec p\cdot\hat r}{r^2}}$$

$$(\vec E_{dipole})_i = -\partial_i V_{dipole}$$

$$= -\frac{1}{4\pi\varepsilon_0}\partial_i\left[\frac{p_j r_j}{r^3}\right]$$

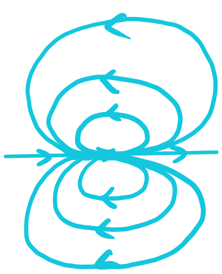
$$= -\frac{p_j}{4\pi\varepsilon_0}\partial_i\left(\frac{r_j}{r^3}\right)$$

$$= -\frac{p_j}{4\pi\varepsilon_0}\left[\frac{\partial_i r_j}{r^3} + r_j\,\partial_i\left(\frac{1}{r^3}\right)\right]$$

$$= -\frac{p_j}{4\pi\varepsilon_0}\left[\frac{\delta_{ij}}{r^3} - 3r_j\frac{1}{r^4}\partial_i r\right]$$

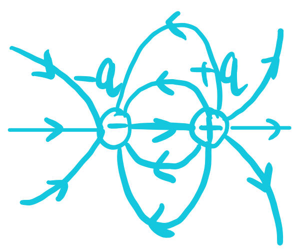
$$= -\frac{p_j}{4\pi\varepsilon_0}\left[\frac{\delta_{ij}}{r^3} - \frac{3r_i r_j}{r^5}\right]$$

$$\vec E_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec p\cdot\hat r)\hat r - \vec p\right]$$



"pure" dipole field

$\nabla\times E = 0$



$\vec p = q\,d\hat e$

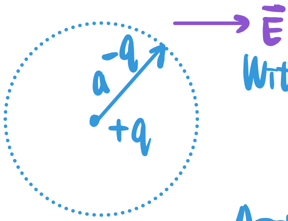
physical dipole

Insulator

Neutron atoms/molecules

intrinsic dipole moment?

Polarization : Induce electric dipole in response to external field



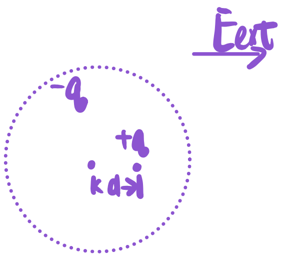
Without External field:

Electrostatic System

Applying External field:

Push positively charged electric cloud to the right

Push negatively charged electric cloud to the left

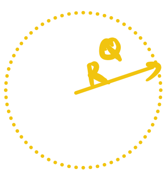


$$F_{ext} = E_{ext}\, q = \frac{q^2}{4\pi\varepsilon_0 a^3} d$$

$$\underbrace{qd}_{\text{dipole moment}} = 4\pi\varepsilon_0 a^3 E_{ext}$$

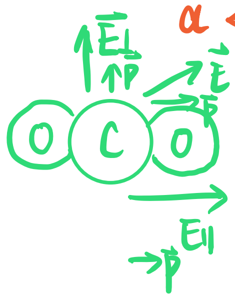
$$\vec{p} = \underbrace{4\pi\varepsilon_0 a^3}_{\alpha} \vec{E}_{ext}$$

$\alpha \leftarrow$ polarizability



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r}{R}\right)^2 \frac{\hat{r}}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 R^2}\hat{r}$$



$$P_i = \alpha_{ij} E_{ext,j}$$

α_{ij} ↑ polarizability tensor

$$\vec F = \int d\tau\, \rho \vec E_{ext} = \vec E_{ext} \int d\tau\, \rho = 0$$

$$F_i = \int d\tau\, \rho(\vec r)\, \vec E_{ext\,i}(\vec r) = \int d\tau\, \rho(\vec r) \left[E_{a_i}(\vec 0) + (\partial_j \left[E_{ext\,i}(\vec 0) + (\partial_j E_{ext}(\vec 0)) r_j + \cdots \right] \right]$$

$$= 0 + [\partial_i E_{ext\,i}(\vec 0)] \underbrace{\int d\tau\, \rho(\vec r)\, r_j}_{p_j} +$$

$$= p_j\, \partial_j E_{ext}(\vec 0)$$

$$\vec F = (\vec p \cdot \vec\nabla) \vec E_{ext} \qquad F_i = p_i\, \partial_i E_{ext\,i}$$

$$\vec N = \int d\tau\, \vec r \times (\rho \vec E_{ext})$$

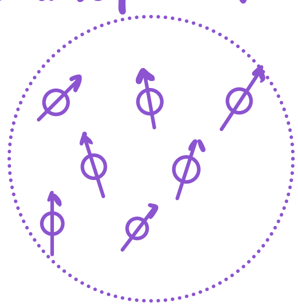
$$= \underbrace{\int d\tau\, \rho \vec r}_{\vec p} \times \vec E_{ext}$$

$$= \vec p \times \vec E_{ext}$$

Macroscopic field

vs.

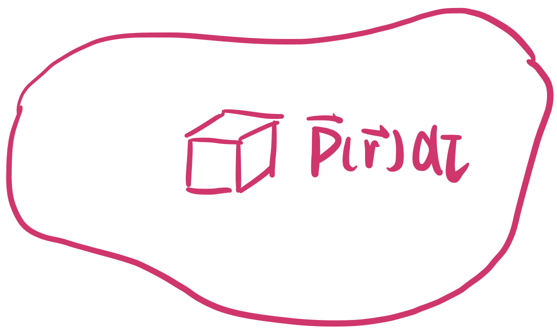
Microscopic field



$$\vec E(\vec r) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec r')}{|\vec r - \vec r'|^3} (\vec r - \vec r')$$

$$\overset{=\vec{E}}{\vec{E}_{ave}}(\vec{r}) = \frac{1}{\frac{4\pi}{3}R^3}\int d\tau' \, \vec{E}_{macro}(\vec{r}+\vec{r}')$$

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_i|^3}\left[\frac{3\vec{p}_i(\vec{r}-\vec{r}_i)(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^2}-\vec{p}_i\right]$$ ← Macroscopic field



Polarization

$$\vec{P} = \frac{\text{total molecular dipole momentum}}{\text{volume}}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d\tau' \, \frac{\vec{P}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\nabla\frac{1}{|\vec{r}-\vec{r}'|} = +\nabla'\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_V d\tau' \, \vec{P}(\vec{r}')\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_V d\tau'\left[\nabla'\cdot\left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\nabla'\vec{P}(\vec{r}')\right]$$

$$= \frac{1}{4\pi\epsilon_0}\oint_S d\vec{a}'\cdot\frac{\vec{P}(\vec{r}')\hat{n}'}{|\vec{r}-\vec{r}'|} - \frac{1}{4\pi\epsilon_0}\int_V d\tau'\frac{\nabla'\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

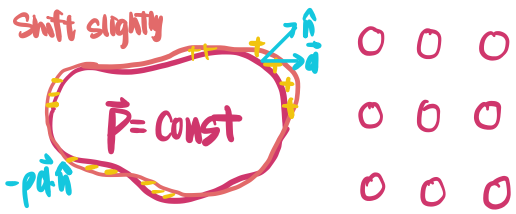
bound surface charge

$\sigma_b \equiv \vec{P}\cdot\hat{n}$

$$= \frac{1}{4\pi\epsilon_0}\oint_S da' \frac{\sigma_b(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\epsilon_0}\int_V d\tau'\frac{\rho_b(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

bound volume charge

$\rho_b = -\vec{\nabla}\cdot\vec{P}$

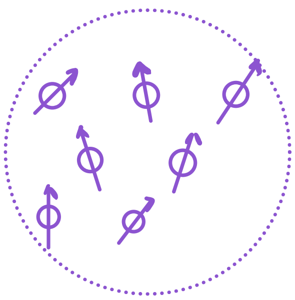


same amount of separation

$\rho_+ + \rho_- = 0$

$\sigma_b = \rho_+ \vec{d}\cdot\hat{n} = \vec{P}\cdot\hat{n}$

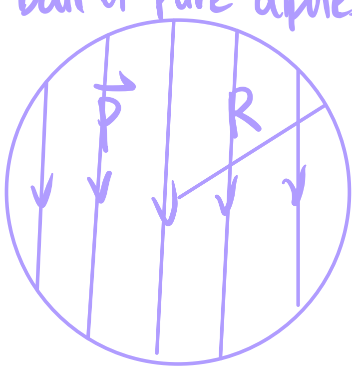
Things only happen on boundary



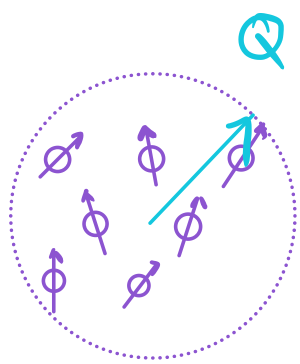
$\vec{E} = \vec{E}_{out} + \vec{E}_{in}$

$$\boxed{\vec{E}_{in} = -\frac{\vec{P}}{3\varepsilon_0}} = -\frac{1}{3\varepsilon_0}\,\frac{3}{4\pi R^3}\sum_{i=1}^{N}\vec{p}_i$$

uniform ball of pure dipoles



$\vec{E}_{in} = -\frac{\vec{P}}{3\varepsilon_0}$



$$Q\vec{E}_{in} = \int d\tau\, \rho(\vec{r})\, \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{R^3}$$

$$= \frac{Q}{4\pi\epsilon_0 R^3}\underbrace{\int d\tau\, \rho(\vec{r})\, \vec{r}}_{\sum p_i = \vec{p}\,\frac{4\pi}{3}R^3} = \frac{Q}{4\pi\epsilon_0 R^3}\,\frac{4\pi}{3}R^3\,\vec{p}$$

$$\vec{E}_{in} = -\frac{\vec{p}}{3\epsilon_0}$$