# First Order Hamitonian perturbation theory

We transform

$$H(\underline{I},\underline{\theta}) = H_0(\underline{I}) + \varepsilon H_1(\underline{I},\underline{\theta}) + O(\varepsilon^2)$$

$$\bar{H}(\bar{\underline{I}},\bar{\underline{\theta}}) = \bar{H}_0(\bar{\underline{I}}) + \varepsilon \bar{H}_1(\bar{\underline{I}}) + O(\varepsilon^2)$$

$$(\underline{I},\underline{\theta}) \to (\bar{\underline{I}},\bar{\underline{\theta}})$$

$$S(\bar{\underline{I}},\underline{\theta}) = \bar{\underline{I}}\cdot\underline{\theta} + \varepsilon S_1(\bar{\underline{I}},\underline{\theta}) + O(\varepsilon^2)$$

$$H_1(\underline{I},\underline{\theta}) = H_{10}(\underline{I}) + \sum_{\underline{m}\neq 0} H_{1\underline{m}}(\underline{I})\, e^{i\underline{m}\cdot\underline{\theta}}$$

$$\underline{\omega}_0(\bar{\underline{I}})\cdot \frac{\partial S_1}{\partial \underline{\theta}}(\bar{\underline{I}},\bar{\underline{\theta}}) = -\sum_{\underline{m}\neq 0} H_{1\underline{m}}(\bar{\underline{I}})\, e^{i\underline{m}\cdot\bar{\underline{\theta}}}$$

Pendulm

$$H_\varepsilon = \omega_0 I - \frac{\varepsilon}{48} I^2 (3 - 4\cos 2\theta + \cos 4\theta) + O(\varepsilon^2)$$

(*) $$\omega_0 \frac{\partial S_1}{\partial \theta} = \frac{\bar{I}^2}{48} (-4\cos 2\bar{\theta} + \cos 4\bar{\theta})$$

$$\bar{H}_0(\bar{I}) = H_0(\bar{I}) = \omega_0 \bar{I}$$

$$\bar{H}_1(\bar{I}, \bar{\theta}) = -\frac{\bar{I}^2}{48} (3 - 4\cos 2\bar{\theta} + \cos 4\bar{\theta})$$

$$+ \omega_0 \frac{\partial S_1}{\partial \theta} (\bar{I}, \bar{\theta})$$

new Hamitonian

$$\bar{H}(\bar{I}) = \omega_0 \bar{I} - \varepsilon \frac{\bar{I}^2}{16} + O(\varepsilon^2)$$

$$\Rightarrow \begin{array}{l} \dot{\bar{I}} = O(\varepsilon^2) \\ \dot{\bar{\theta}} = \omega_0 - \frac{\varepsilon \bar{I}}{8} + O(\varepsilon^2) = \omega(\bar{I}) \end{array}$$

Lie perturbative theory (time-independent case)

Consider a canonical transformation

$\bar{z} = Z(z)$ Associative with $Z$ is

a transformation

operator $T$

$(TF)(\underline{z}) = F(\underline{\bar{z}}(\underline{z}))$

$T\underline{z} = \underline{\bar{z}}(\underline{z}) = \underline{\dot{z}}$

$\bar{z} = (q,p)$

In particular

$K(\bar{\bar{z}}) = K(Z(\bar{z})) = H(z)$

$K(\bar{\bar{z}}) = H(Z^{-1}(\bar{z}))$

$K = T^{-1}H \quad (*)$

The canonical transformation

is generated by

Infinitesimal canonical transformation

generated by generator $G(z,\varepsilon)$

In terms of $G$

$T = \exp - \int d\varepsilon L_G$

$L_G F = \{G, F\}$

Suppose

$G(z,\varepsilon) = G_1(z) + \varepsilon G_2(z) + \varepsilon^2 G_3(z)$

$L_{G_i} = L_i$

$T = 1 - \varepsilon L_1 + \frac{1}{2}\varepsilon^2(-L_2 + L_1^2) + O(\varepsilon^3)$

$T^{-1} = 1 + \varepsilon L_1 + \frac{1}{2}\varepsilon^2(L_2 + L_1^2) + O(\varepsilon^3)$

Suppose $K = \sum_{n=0}^{\infty} \varepsilon^n K_n$

$H = \sum_{n=0}^{\infty} \varepsilon^n H_n$

and substitute $T^{-1}$ into $K = T^{-1}H$

$O(\varepsilon^0)$ $K_0 = H_0$

$K_0(\bar{z}) = H_0(\bar{z})$

$O(\varepsilon)$ : $K_1 = H_1 + L_1 H_0$

$O(\varepsilon^2)$ : $K_2 = H_2 + L_1 H_1 + \frac{1}{2}(L_2 + L_1^2) H_0$

$O(\varepsilon^0)$ : $K_0(\bar{z}) = H_0(\bar{z})$

$O(\varepsilon^1)$ : $L_1 H_0 = \{G_1, H_0\} = \left(\frac{\partial G_1}{\partial z}\right)^T \underline{\underline{J}} \cdot \frac{\partial H_0}{\partial z} \equiv D_0 G_1$

$\left( \text{where } D_0 G_1 = \frac{\partial G_1}{\partial t} + \{G_1, H_0\} \right)$

$\dot{q} = \{q, H\}$ $\dot{p} = \{p, H\}$

$D_0$ time derivative

$O(\varepsilon^0)$ $\quad 0 = K_0 - H_0$

$O(\varepsilon)$ $\quad D_0 G_1 = K_1 - H_1$

$O(\varepsilon^2)$ $\quad D_0 G_2 = 2(K_2 - H_2) - L_1(H_1 + K_1)$

ie $D_0 G_1 = K_1 - H_1$ is a differential eq. for $G_1$ and $K_1$

Similarly

Example : Pendulum Problem

$H_\varepsilon(I,\theta) = H_0 + \varepsilon H_1 + O(\varepsilon^2)$

$H_0(I,\theta) = \omega_0 I$

$H_1(I,\theta) = -\frac{1}{48} I^2 (3 - 4\cos 2\theta + \cos 4\theta)$

$O(\varepsilon^0) = K_0(\bar{I}) = \omega_0 \bar{I}$

$O(\varepsilon^1) = K_1 = H_1 + D_0 G = H_1 + \{G_1, H_0\} = H_1 + \omega_0 \frac{\partial G_1}{\partial \theta}$

$\omega_0 \frac{\partial G_1}{\partial \theta} = K_1 - H_1$ suppose we try $K_1 = 0$

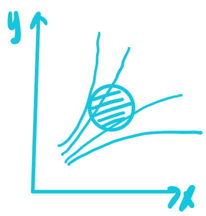
$$A^{\mu} = \begin{pmatrix} \phi \\ \bar{A} \end{pmatrix}$$

$$F^{\mu\nu} = \partial^{\mu} A^{\mu} - \partial^{\nu} A^{\nu} = \left(\begin{array}{c|c} 0 & E^{j} \\ \hline -E^{i} & \epsilon^{ijk} B^{k} \end{array}\right)$$

$$\frac{dp^{\mu}}{d\tau} = q F^{\mu\nu} u_{\nu}$$

$$u^{\nu'} = \begin{pmatrix} \gamma \\ \gamma \bar{v} \end{pmatrix}$$

Fluid




$\delta V(x)$

$N_q(x)$

$P(x)$

$\bar{v}(x)$

$T(x)$

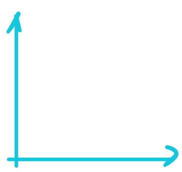
$$n_i(x) = \frac{N_i}{\delta V}$$

$i$ carry $q_i$

$$\rho_i(x) = n_i(x) q_i$$

$$\rho = \sum_i \rho_i = \sum_i n_i q_i$$

$$\bar{J}(x) = \rho \bar{v} = \sum_i n_i q_i \bar{v}$$



$$Q = \int_V \rho \, d^3x$$