Derivation of Hamiltonian

Virtual Work

$$\delta W = \sum_i f_i \delta x_i = \sum_i \sum_\alpha f_i \frac{\partial x_i}{\partial q_\alpha} \delta q_\alpha = \sum_\alpha Q_\alpha \delta q_\alpha$$

$f_i = F_i + N_i$

↑ constrain force

$$Q_\alpha = \sum_i f_i \frac{\partial x_i}{\partial q_\alpha}$$

$$\delta W = \sum_i F_i \delta x + \sum_i N_i \delta x_i$$

Condition $\sum_i N_i \delta x_i = 0$

$$T = \sum_i \frac{1}{2} m_i \dot{x}_i^2$$

$$x_i = x_i(q_1, \dots, q_n, t)$$

$$\dot{x}_i = \sum_\alpha \frac{\partial x_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial x_i}{\partial t}$$

$$\frac{\partial T}{\partial \dot{q}_\alpha} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_\alpha}$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_\alpha} = \frac{\partial x_i}{\partial q_\alpha}$$

$$= \sum_i m_i \dot{x}_i \frac{\partial x_i}{\partial q_\alpha}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) = \sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\alpha} + \sum_i m_i \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\alpha}\right)$$

$$= \sum_i \left[(F_i + N_i) \frac{\partial x_i}{\partial q_\alpha} + m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_\alpha} \right]$$

$$= \sum_i F_i \frac{\partial x_i}{\partial q_\alpha} + \frac{\partial}{\partial q_\alpha}\left(\sum_i \frac{1}{2} m_i \dot{x}_i^2\right) = Q_\alpha + \frac{\partial T}{\partial q_\alpha}$$

$$\frac{\partial T}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) + Q_\alpha = 0$$

For conservative force of F_i

$$F_i = -\frac{\partial V}{\partial x_i}$$

$$Q_\alpha = \sum_i F_i \frac{\partial x_i}{\partial q_\alpha} = -\sum_i \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$$

$$\frac{\partial T}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - Q_\alpha = 0$$

$$\Rightarrow \frac{\partial T}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) + \frac{\partial V}{\partial q_\alpha} = 0$$

U does not depend on $\dot{q}_\alpha$

$$L(q_\alpha, \dot{q}_\alpha, t) \equiv T - U \qquad \frac{\partial T}{\partial \dot{q}_\alpha} = \frac{\partial L}{\partial \dot{q}_\alpha}$$

[Potential Energy does not depend on velocity.]

$$\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$$

7.4 Hamiltonian of a system

$L = L(q_\alpha, \dot{q}_\alpha, t)$

$q_\alpha, \dot{q}_\alpha, L \Rightarrow q_\alpha, p_\alpha, H(q_\alpha, p_\alpha, t)$

$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$ as generalized momentum $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}$

$\Downarrow$

express $\dot{q}_\alpha$ in terms of p_α, t

* Special Case $L = \frac{1}{2}m\dot{x}^2 - U(x)$ $\qquad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$\dot{x} = \frac{p}{m}$$

$$H(q_\alpha, p_\alpha, t) = -\left(L(q_\alpha, \dot{q}_\alpha, t) - \sum_\alpha p_\alpha \dot{q}_\alpha\right)$$

$$= \sum_\alpha p_\alpha \dot{q}_\alpha - L$$

Special Case $L = \frac{1}{2}m\dot{x}^2 - U(x)$

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$

$$H = P\dot{x} - \left[\tfrac{1}{2}m\dot{x}^2 - U(x)\right]$$

$$= P\left(\frac{P}{m}\right) - \frac{1}{2}m\left(\frac{P}{m}\right)^2 + U(x)$$

$$= \frac{P^2}{m} - \frac{P^2}{2m} + U(x)$$

$$= \frac{P^2}{2m} + U(x)$$

$$= \frac{1}{2}m\dot{x}^2 + U(x)$$

$$= T(x) + U(x)$$

$$= \text{Total Energy}$$

$$H = \sum_\alpha P_\alpha \dot{q}_\alpha - L$$

$$H + L = \sum_\alpha P_\alpha \dot{q}_\alpha$$

$$H(q_\alpha, p_\alpha, t) + L(q_\alpha, \dot{q}_\alpha, t) = \sum_\alpha P_\alpha \dot{q}_\alpha$$

Euler's Equation $\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$

Euler's 2nd Equation $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_\alpha \left[\frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_\alpha}\right]\right]$

$\frac{\partial L}{\partial q_\alpha} = 0$

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L$$

$$\begin{cases} x_1 = x_1(q_1, \dots, q_s, t) \\ x_2 = x_2(q_1, \dots, q_s, t) \\ \vdots \\ x_n = x_n(q_1, \dots, q_s, t) \end{cases}$$

If $\frac{\partial x_i}{\partial t} = 0$ then $H = T + U$

Equation of Motion

$$H(q_\alpha, p_\alpha, t) + L(q_\alpha, \dot{q}_\alpha, t) = \sum_\alpha p_\alpha \dot{q}_\alpha$$

1) $\frac{\partial H}{\partial q_\alpha} + \frac{\partial L}{\partial q_\alpha} = 0 \quad \frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{q}_\alpha}}_{p_\alpha}\right) = 0 \Rightarrow \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$

$$\frac{dp_\alpha}{dt} = \frac{\partial L}{\partial q_\alpha} = -\frac{\partial H}{\partial q_\alpha}$$

2) $\frac{\partial H}{\partial p_\alpha} + 0 = \dot{q}_\alpha$

$$\boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \qquad \dot{q}_\alpha = +\frac{\partial H}{\partial p_\alpha}}$$

$H(q_\alpha, p_\alpha, t)$

$$\frac{dH}{dt} = \sum_\alpha \left(\frac{\partial H}{\partial q_\alpha} \underbrace{\dot{q}_\alpha}_{\frac{\partial H}{\partial p_\alpha}} + \frac{\partial H}{\partial p_\alpha} \underbrace{\dot{p}_\alpha}_{-\frac{\partial H}{\partial q_\alpha}} \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\alpha \left(\frac{\partial H}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha} \right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Any $f(q_\alpha, p_\alpha, t)$

$$\frac{df}{dt} = \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha \right) + \frac{\partial f}{\partial t}$$

$$= \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} + \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha} \right) + \frac{\partial f}{\partial t}$$

Poisson Bracket

$$U = U(q_\alpha, p_\alpha, t) \qquad V = V(q_\alpha, p_\alpha, t)$$

$$[U,V] = \sum_\alpha \left(\frac{\partial U}{\partial q_\alpha}\frac{\partial V}{\partial p_\alpha} - \frac{\partial U}{\partial p_\alpha}\frac{\partial V}{\partial q_\alpha} \right)$$

$$= \sum_\alpha \det \begin{vmatrix} \frac{\partial U}{\partial q_\alpha} & \frac{\partial U}{\partial p_\alpha} \\ \frac{\partial V}{\partial q_\alpha} & \frac{\partial V}{\partial p_\alpha} \end{vmatrix}$$

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}$$