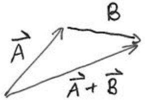
Vector Analysis

scalar: magnitude, no direction Ex: mass, charge, density, temperatu
vector: magnitude and direction Ex: displacement, velocity
linear momentum, angular momentum
E/B fields

Printed form: boldface $\mathbf{A}$
Handwritten form: $\vec{A}$
magnitude: $A = |\vec{A}|$

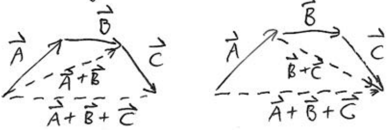
"geometric" representation

Vector Addition



commutativity $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

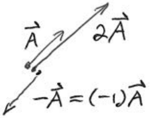
associativity $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$



Multiplication by a scalar

$a\vec{A}$

(a is a real number)



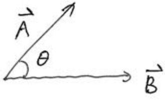
distributivity $(a+b)\vec{A} = a\vec{A}+b\vec{A}$

$a(\vec{A}+\vec{B}) = a\vec{A}+a\vec{B}$

"zero" vector $0\vec{A}=\vec{0}$

~~magnitude~~ unit vector: $\hat{A} = \vec{A}/|\vec{A}| = \vec{A}/A$

Dot product (Scalar product)



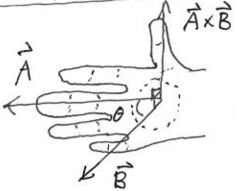
$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta}$$

acute angle?
obtuse angle?
right angle?

symmetric $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

distributivity: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

Cross product (vector product)



$$\boxed{\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n},}\quad \hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

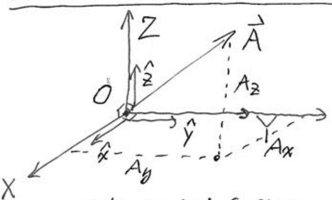
direction of $\hat{n}$ determined by the right-hand rule!

anti-symmetry $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (or simply)

cross product with self: $\vec{A}\times\vec{A} = \vec{0}$, $\vec{A}\times\vec{A} = 0$

distributivity $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

Cartesian Coordinate system



right-handed system

Unit "basis" vectors: $\hat{x}$, $\hat{y}$, $\hat{z}$

Orthonormality

$$\hat{x}^2 = \hat{y}^2 = \hat{z}^2 = 1$$

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$$

cross products

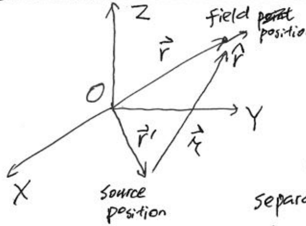
$$\hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y}$$

Cartesian representation

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$A_x = \vec{A} \cdot \hat{x}, \quad A_y = \vec{A} \cdot \hat{y}, \quad A_z = \vec{A} \cdot \hat{z}$$

Position, displacement, separation vectors



position: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

radius $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

unit vector $\hat{r} = \vec{r}/r$ (Pythagoras theorem)

separation: vector:

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$\mathscr{r} = |\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

infinitesimal displacement

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

(or $d\vec{r}$)

Indicial Notation

Cartesian components $\vec{A}$: A_i, $i = 1, 2, 3$

Kronecker delta: $\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$

Dot product

$$\vec{A} \cdot \vec{B} = \sum_i \sum_j \delta_{ij} A_i B_j = \sum_i A_i B_i$$

(anti-symmetric) Levi-Civita Symbol: ε_{ijk}

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two indices the same} \\ 1, & \text{if } i,j,k = 1,2,3 \text{ or cyclic permutations} \\ -1, & \text{if } i,j,k = 2,1,3 \text{ or cyclic permutations} \end{cases}$$

$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ ······ even

$(i, j, k) = (2, 1, 3), (1, 3, 2), (3, 2, 1)$ ······ odd

cross product

$$(\vec{A} \times \vec{B})_i = \sum_j \sum_k \varepsilon_{ijk} A_j B_k = -\sum_j \sum_k \varepsilon_{ijk} B_j A_k$$

Einstein's Summation convention

dummy indices

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \rightarrow A_i B_i$$

$$(\vec{A} \times \vec{B})_i = \sum_j \sum_k \varepsilon_{ijk} A_j B_k \rightarrow \varepsilon_{ijk} A_j B_k$$

free index

More examples:

$$\vec{A}(\vec{B} \cdot \vec{C}) \longrightarrow A_i B_j C_j$$

Einstein's summation convention: Examples

scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i(\vec{B}\times\vec{C})_i = A_i\,\varepsilon_{ijk}B_jC_k = \varepsilon_{ijk}A_iB_jC_k$$

cyclic symmetry:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

matrix determinant

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

vector triple product

use $\boxed{\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$

$$\left[\vec{A}\times(\vec{B}\times\vec{C})\right]_i = \varepsilon_{ijk}A_j(\vec{B}\times\vec{C})_k = \varepsilon_{ijk}\varepsilon_{klm}A_jB_lC_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m = A_jB_iC_j - A_jB_jC_i$$

so $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

Discussion

① $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$, ~~reduce $\nabla(\vec{A}\cdot\vec{B})$~~

Reduce the expression using Einstein's summation convention

② How vector / tensors transform under rotations ?
(1.1.5)

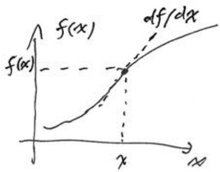
③ $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$

Differential Calculus

Ordinary derivative

$$df = \left(\frac{df}{dx}\right) dx$$

$$f = f(x)$$



Gradient of a scalar field $f = f(\vec{r}) = f(x, y, z)$

for small $\Delta x, \Delta y, \Delta z$

$$\Delta f = a_1 \Delta x + a_2 \Delta y + a_3 \Delta z + (\text{higher orders})$$

In the limit $\Delta x, \Delta y, \Delta z \to 0$, a_1, a_2, a_3 are partial derivatives

$$\boxed{df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz}$$

$\dashrightarrow$ $df = (\nabla f) \cdot d\vec{l}$, where the vector

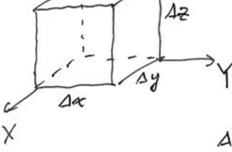
$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \quad \cdots \text{gradient}$$

The "del" operator $\quad \nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

Divergence of a vector field $\vec{v}(\vec{r}) = \vec{v}(x, y, z)$

(i.e. $v_i = v_i(x, y, z)$)

$$\boxed{\nabla \cdot \vec{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}$$



X: $\left(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}\right)$

Y: $\left(y - \frac{\Delta y}{2}, y + \frac{\Delta y}{2}\right)$

Z: $\left(z - \frac{\Delta z}{2}, z + \frac{\Delta z}{2}\right)$

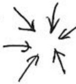
Net flux:

$$\Delta y \Delta z \left[v_x\left(x + \frac{\Delta x}{2}, y, z\right) - v_x\left(x - \frac{\Delta x}{2}, y, z\right) \right] \simeq \Delta x \Delta y \Delta z \frac{\partial v_x}{\partial x}$$

Account for all 3 pairs of faces facing all three directions

$$\text{"flux"} = \underbrace{\Delta x \Delta y \Delta z}_{\text{"volume"}} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

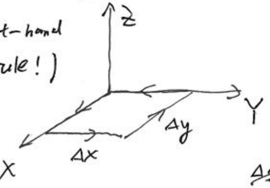
so $\quad \text{divergence} = \lim_{\text{volume} \to 0} \dfrac{\text{net flux}}{\text{volume}}$



~~Example:~~ Curl of a vector field

$$\boxed{\nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}}$$

(right-hand rule!)



$X: \left(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2} \right)$

$Y: \left(y - \frac{\Delta y}{2}, y + \frac{\Delta y}{2} \right)$

"circulation"

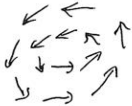
$$\cancel{\Delta x \left[v_y\left(y + \frac{\Delta y}{2}\right) - v_y\left(y - \frac{\Delta y}{2}\right) \right] \simeq \Delta x \Delta y}$$

$$\Delta x \left[v_y\left(x + \frac{\Delta x}{2}, y, z\right) - v_y\left(x - \frac{\Delta x}{2}, y, z\right) \right] \simeq \Delta x \Delta y \frac{\partial v_y}{\partial x}$$

$$\Delta y \left[-v_x\left(x, y + \frac{\Delta y}{2}, z\right) + v_x\left(x, y - \frac{\Delta y}{2}, z\right) \right] \simeq -\Delta x \Delta y \frac{\partial v_x}{\partial y}$$

Total circulation around z axis $\simeq \Delta x \Delta y \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \right)$

$$\text{curl} = \lim_{\text{area} \to 0} \frac{\text{circulation}}{\text{area}} \leftarrow \text{(along all 3 directions)}$$



Product rules for ~~vector~~ differential calculus

Function of 1 variable

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \quad , \quad \frac{d}{dx}(kf) = k\frac{df}{dx}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f\frac{dg}{dx} - \frac{df}{dx}g}{g^2}$$

Similarly for gradient, div., curl.

$$\nabla(f+g) = \nabla f + \nabla g \, , \quad \nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$

$$\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$$

$$\nabla(kf) = k\nabla f \, , \quad \nabla\cdot(k\vec{A}) = k\nabla\cdot\vec{A} \, , \quad \nabla\times(k\vec{A}) = k(\nabla\times\vec{A})$$

Products

$$\nabla(fg) = g(\nabla f) + f(\nabla g)$$

$$\nabla\cdot(f\vec{A}) = (\nabla f)\cdot\vec{A} + f(\nabla\cdot\vec{A})$$

$$\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f(\nabla\times\vec{A})$$

Now apply Einstein's summation to work out a few other rules

① $\nabla\cdot(\vec{A}\times\vec{B}) = \varepsilon_{ijk}\,\partial_i(A_j B_k) = \varepsilon_{ijk}\left[(\partial_i A_j)B_k - A_j(\partial_i B_k)\right]$

$$= \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

② $\left[\nabla\times(\vec{A}\times\vec{B})\right]_i = \varepsilon_{ijk}\,\partial_j(\vec{A}\times\vec{B})_k = \varepsilon_{ijk}\,\varepsilon_{klm}\,\partial_j(A_l B_m)$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(B_m\partial_j A_l + A_l\partial_j B_m)$$

$$= B_j\partial_j A_i - B_i\partial_j A_j + A_i\partial_j B_j - A_j\partial_j B_i = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

Second derivatives

(1) Div. of grad. $\nabla\cdot(\nabla f) \equiv \nabla^2 f$

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad (\nabla^2 \text{ is the Laplacian})$$

(2) Curl of grad.

symmetric $\partial_j\partial_k f = \partial_k\partial_j f$

$$\nabla\times(\nabla f) = 0$$

because $[\nabla\times(\nabla f)]_i = \varepsilon_{ijk}\,\overbrace{\partial_j\,\partial_k}\, f = 0$

(3) double curl:

$$\left[\nabla\times(\nabla\times\vec{A})\right]_i = \varepsilon_{ijk}\,\partial_j(\nabla\times\vec{A})_k = \varepsilon_{ijk}\,\varepsilon_{klm}\,\partial_j\partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l A_m = \partial_j\partial_i A_j - \partial_j\partial_j A_i$$

$$= \left[\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}\right]_i$$

So $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

(4) divergence of curl

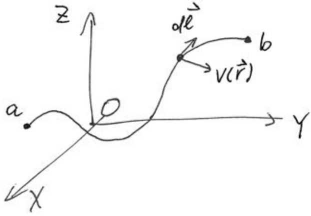
$$\nabla\cdot(\nabla\times\vec{A}) = \partial_i(\nabla\times\vec{A})_i = \partial_i(\varepsilon_{ijk}\,\partial_j A_k)$$

$$= \varepsilon_{ijk}\,(\underbrace{\partial_i\partial_j}_{\text{symmetric}} A_k) = 0$$

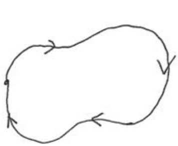
so $\nabla\cdot(\nabla\times\vec{A}) = 0$

Integral Calculus

(a) line integral



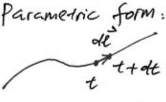
$$\int_a^b \vec{V}(\vec{r})\cdot d\vec{\ell}$$



$$\oint \vec{V}(\vec{r})\cdot d\vec{\ell}$$ ...... integration along a closed loop

~~closed loop~~

Parametric form:

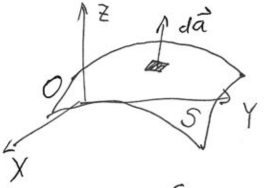


$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

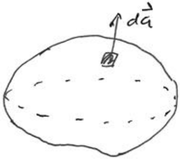
$$d\vec{\ell} = dt\left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right)$$

$$\int_a^b \vec{V}(\vec{r})\cdot d\vec{\ell} = \int_{t_a}^{t_b} dt\left[V_x\frac{dx}{dt} + V_y\frac{dy}{dt} + V_z\frac{dz}{dt}\right] \longleftarrow \text{1D integral}$$

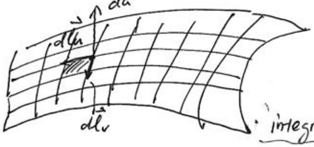
(b) Surface integral



$$\int_S \vec{V}(\vec{r})\cdot d\vec{a}$$



$$\oint \vec{V}(\vec{r})\cdot d\vec{a}$$ ...... integration over closed surface



$d\vec{\ell}_u$, $d\vec{\ell}_v$, $d\vec{a}$ form right-handed system

Parametric form:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$$d\vec{\ell}_u = du\left(\frac{\partial x}{\partial u}\hat{x} + \frac{\partial y}{\partial u}\hat{y} + \frac{\partial z}{\partial u}\hat{z}\right)$$

$$d\vec{\ell}_v = dv\left(\frac{\partial x}{\partial v}\hat{x} + \frac{\partial y}{\partial v}\hat{y} + \frac{\partial z}{\partial v}\hat{z}\right)$$

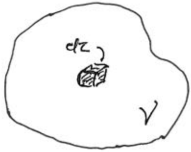
$$d\vec{a} = d\vec{\ell}_u \times d\vec{\ell}_v$$

integrand for surface integral

$\vec{V}(\vec{r})\cdot(d\vec{\ell}_u \times d\vec{\ell}_v)$ (2D integral)

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ V_x & V_y & V_z \end{vmatrix} du\,dv$$

(c) Volume Integral



$$\int_V T(\vec{r})\, d\tau$$

$T = T(x, y, z)$, $d\tau = dx\, dy\, dz$

$$\int_V T(\vec{r})\, d\tau = \iiint_V dx\, dy\, dz\; T(x, y, z)$$

(3D integral)

Fundamental Theorem of Calculus

Simple Example:

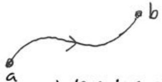
$$(1-2)+(2-3)+(3-4)+\cdots\cdots+(98-99)+(99-100)$$

$$= 1-100 = -99$$

* Fundamental theorem of calculus for functions of 1 variable

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

* Fundamental theorem for gradient



$$\int_a^b (\nabla T)\cdot d\vec{l} = T(b) - T(a)$$

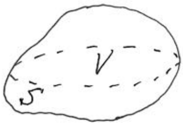
independent of path!

single-valued $T(\vec{r})$

$$\oint (\nabla T)\cdot d\vec{l} = 0$$

* Green's Theorem

$$\int_V (\nabla\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$$



* Stokes' Theorem

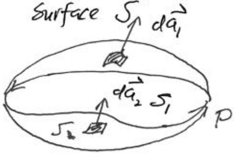
surface orientation: right-hand rule!

$$\int_S (\nabla\times\vec{V})\cdot d\vec{a} = \oint_P \vec{V}\cdot d\vec{l}$$

$$\oint_S (\nabla\times\vec{V})\cdot d\vec{a} = 0\,, \quad \text{if } S \text{ is a closed surface}$$

$$\oint_P \vec{v}\cdot d\vec{l} = \int_S (\nabla\times\vec{V})\cdot d\vec{a}$$

independent of choice for the surface S

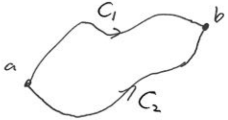


$$\int_{S_1} (\nabla\times\vec{V})\cdot d\vec{a} - \int_{S_2} (\nabla\times\vec{V})\cdot d\vec{a}$$

$$= \oint_{S_1+\bar{S}_2} (\nabla\times\vec{V})\cdot d\vec{a} = \int_V \underbrace{\nabla\cdot(\nabla\times\vec{V})}_{0}\, d\tau = 0$$

Green's Theorem

$$T(b) - T(a) = \int_C^{b} (\nabla T)\cdot d\vec{l}$$



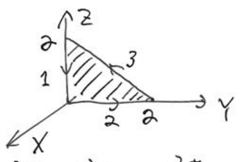
$$\int_{C_1} (\nabla T)\cdot d\vec{l} - \int_{C_2} (\nabla T)\cdot d\vec{l}$$

$$= \oint_{C=C_1+\bar{C}_2} (\nabla T)\cdot d\vec{l} = \int_S \underbrace{[\nabla\times(\nabla T)]}_{0}\cdot d\vec{a} = 0$$

Stokes' theorem

independent of choice for the path C

Test Stokes' theorem



$$\vec{V} = (xy)\,\hat{x} + (2yz)\,\hat{y} + (3zx)\,\hat{z}$$

$$\nabla \times \vec{V} = (-2y)\,\hat{x} + (-3z)\,\hat{y} + (x)\,\hat{z}$$

Surface integral $= \int_0^2 dy \int_0^{2-y} dz\,(-2y) = \int_0^2 dy\,(-2y)\int_0^{2-y} dz$

$$= -2\int_0^2 dy\; y(2-y) = -2\left(4 - \frac{1}{3}2^3\right) = -\frac{8}{3}$$

line integral 1 $= \int_2^0 (-dz)(3\cdot z\cdot 0) = 0$

line integral 2 $= \int_0^2 dy\,(2\cdot y\cdot 0) = 0$

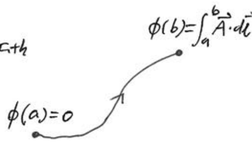
line integral 3 $= \int_2^0 dy\,(\hat{y} - \hat{z})\cdot\vec{V} = \int_2^0 dy\,\left(+2y(2-y) - 3(2-y)0\right)$

$(z = 2-y)$

$$= -2\int_0^2 dy\; y(2-y) = -2\left[(2^2 - 0^2) - \frac{1}{3}(2^3 - 0^3)\right] = -\frac{8}{3} \quad \checkmark$$

---

* Curlless field (irrotational)



$\nabla \times \vec{A} = 0 \iff \int_a^b \vec{A}\cdot d\vec{l}$ independent of path

$\iff \oint \vec{A}\cdot d\vec{l} = 0 \iff \vec{A} = \nabla\phi$

---

* divergence-free field (solenoidal)

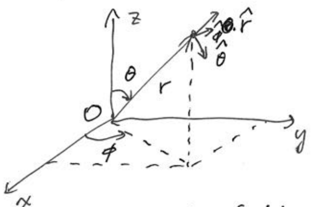
$\nabla\cdot\vec{A} = 0 \iff \int_S \vec{A}\cdot d\vec{a}$ independent of surface S, given boundary P

$\iff \oint_S \vec{A}\cdot d\vec{a} = 0 \iff \vec{A} = \nabla\times\vec{B}$ Can explicitly construct ϕ and $\vec{B}$

Curvilinear Coordinates

Spherical Coordinates

~~r radius~~



radius: $0 \le r < +\infty$

polar angle: $0 \le \theta \le \pi$

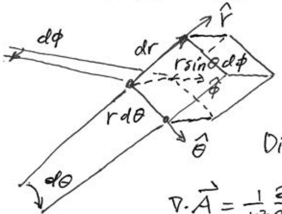
azimuthal angle: $0 \le \phi < 2\pi$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

vector field $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

line element $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

volume element $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$



Grad: $\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\phi}$

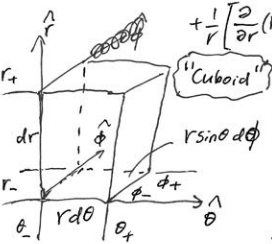
Divergence: $\cancel{\nabla\cdot\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)}$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Laplacian: $$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

Curl: $$\nabla\cdot\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta}$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

(see Appendix A for more)



"Cuboid" flux $= dr\frac{\partial}{\partial r}(r\,d\theta\,r\sin\theta\,d\phi\,A_r)$

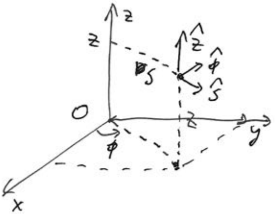
$+\, d\theta\frac{\partial}{\partial\theta}(dr\,r\sin\theta\,d\phi\,A_\theta)$

$+\, d\phi\frac{\partial}{\partial\phi}(dr\,r\,d\theta\,A_\phi)$

volume $= dr\,r\,d\theta\,r\sin\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$

div. = flux / volume

Cylindrical Coordinates



distance to z-axis : $0 \le s < +\infty$

azimuthal angle : $0 \le \phi < 2\pi$

vertical distance : $-\infty < z < +\infty$

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases}$$

vector field: $\vec{A} = A_s\,\hat{s} + A_\phi\,\hat{\phi} + A_z\,\hat{z}$

line element: $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

volume element: $d\tau = s\,ds\,d\phi\,dz$

Grad: $\nabla T = \dfrac{\partial T}{\partial s}\hat{s} + \dfrac{1}{s}\dfrac{\partial T}{\partial \phi}\hat{\phi} + \dfrac{\partial T}{\partial z}\hat{z}$

Divergence: $\nabla\cdot\vec{A} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(sA_s) + \dfrac{1}{s}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}$

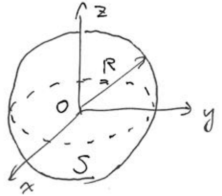
Laplacian: $\nabla^2 T = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial T}{\partial s}\right) + \dfrac{1}{s^2}\dfrac{\partial^2 T}{\partial \phi^2} + \dfrac{\partial^2 T}{\partial z^2}$

Curl: $$\nabla\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{z}$$

Consider the vector field

$$\vec{V} = \frac{1}{r^2}\hat{r}, \quad V_r = \frac{1}{r^2}, \quad V_\theta = V_\phi = 0$$

Divergence $\nabla\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$



$$\int_V \nabla\cdot\vec{V}\, d\tau = \oint_S \vec{V}\cdot d\vec{a} = 0$$

$d\vec{a} = R\,d\theta\; R\sin\theta\, d\phi\; \hat{r}$ evaluate at $r=R$

$$\oint_S \vec{V}\cdot d\vec{a} = \int_0^\pi d\theta\,\sin\theta\int_0^{2\pi} d\phi\left(R^2\hat{r}\cdot\frac{1}{R^2}\hat{r}\right)$$

$$= \int_0^\pi d\theta\,\sin\theta\int_0^{2\pi} d\phi = 4\pi \neq 0 \quad ?!$$

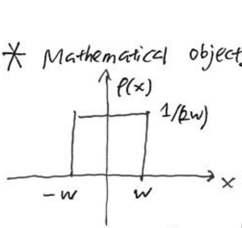
We are missing something: $\nabla\cdot\left(\frac{1}{r^2}\hat{r}\right) = 0$ invalid at $r=0$

In fact $\boxed{\nabla\cdot\left(\frac{1}{r^2}\hat{r}\right) = 4\pi\,\delta^{(3)}(\vec{r})}$ ← Dirac δ function

$$\delta_D^{(3)}(\vec{r}) = \begin{cases} 0, & \text{if } \vec{r}\neq 0 \\ \text{infinity}, & \text{if } \vec{r}=0 \end{cases}, \quad \int_V d\tau\, \delta^{(3)}(\vec{r}) = 1$$

(V is any volume that includes $\vec{r}=0$)

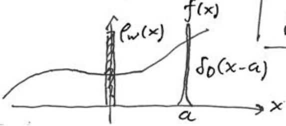
* Mathematical objects with infinite density: point, line, surface, etc.



$$\rho_w(x) = \begin{cases} \frac{1}{2w}, & |x|<w \\ 0, & |x|>w \end{cases}$$

Total mass $= \int \rho_w(x)\,dx = 1$

$$\boxed{\lim_{w\to 0}\rho_w(x) = \delta_D(x)}$$



$$\int dx\, f(x)\rho_w(x) \simeq \int dx\, f(0)\rho_w(x) = f(0) \text{ if } w^{\text{small}}$$

Therefore $\quad \int dx\, \delta_D(x) f(x) = f(0)$ $\qquad$ Q: what's the unit of $\delta_D(x)$?

$$\int dx\, \delta_D(x-a) f(x) = f(a)$$

Property $\quad \boxed{\delta_D(kx) = \frac{1}{|k|}\delta_D(x)}$, in particular $\delta_D(-x) = \delta_D(x)$

if $k>0$ $\quad \int_{-\infty}^{+\infty} dx\, \delta_D(kx) f(x) = \int_{-\infty}^{+\infty} \frac{dy}{k}\delta_D(y) f\left(\frac{y}{k}\right) = \frac{1}{k} f(0)$

$$= \int_{-\infty}^{+\infty} dx\, \frac{1}{k}\delta_D(x) f(x)$$

$$\int_{-\infty}^{+\infty} dx\, \delta_D(-kx) f(x) = \int_{+\infty}^{-\infty} \frac{dy}{-k}\delta_D(y) f\left(\frac{y}{-k}\right) = \frac{1}{k}\int_{-\infty}^{+\infty} dy\, \delta_D(y) f\left(\frac{y}{-k}\right)$$

$$= \frac{1}{k} f(0) = \int_{-\infty}^{+\infty} dx\, \frac{1}{k}\delta_D(x) f(x)$$

Combining both cases: $\quad \delta_D(kx) = \frac{1}{|k|}\delta_D(x)$

Generalization to 3D:

$$\delta_D^{(3)}(\vec r) = \delta_D(x)\delta_D(y)\delta_D(z), \quad \int_V \delta_D^{(3)}(\vec r) f(\vec r)\, d\tau = f(\vec 0)$$

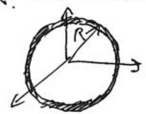
(V includes the point $\vec r = \vec 0$)

Ex: Density of a point mass m at $\vec r = \vec a$

$$\rho(\vec r) = m\, \delta_D^{(3)}(\vec r - \vec a)$$

Ex: ~~charge~~ density of a thin sphere of radius R and total mass M

$$\rho(\vec r) = \frac{1}{4\pi R^2} M \delta_D(r-R)$$



$$\int \rho(\vec r)\, d\tau$$

$$= \int_0^\infty dr \int_0^\pi d\theta\, r \int_0^{2\pi} d\phi\, r\sin\theta\, \rho(\vec r) = \frac{1}{4\pi}\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\frac{M}{R^2}\int_0^\infty dr\, \delta_D(r-R) r^2$$

$$= \frac{1}{4\pi}(4\pi)\frac{M}{R^2}R^2 = M \quad \checkmark$$

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r} = -\frac{1}{r^2}\hat{r}$$

$$\frac{1}{r^2}\hat{r} = -\nabla\left(\frac{1}{r}\right)$$

Laplacian of 1/r potential

$$\nabla\cdot\left(\frac{1}{r^2}\hat{r}\right) = -\nabla^2\left(\frac{1}{r}\right) \Rightarrow \boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta_D^{(3)}(\vec{r})}$$

Helmholtz Theorem

~~Tech~~ vector field $\vec{F}(\vec{r})$

Given divergence & curl $\nabla\cdot\vec{F}(\vec{r}) = D(\vec{r})$

$\nabla\times\vec{F}(\vec{r}) = \vec{C}(\vec{r})$

(curl-free)

$\vec{F} = -\nabla U + \nabla\times\vec{W}$ decomposition into gradient and curl field

(solenoid)

<u>Technical assumption</u>: $\vec{F}(\vec{r})$ vanishes faster than $1/|\vec{r}|$ as $|\vec{r}|\to\infty$

$$\boxed{\delta_D^{(3)}(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi}\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|}}$$

$$\boxed{\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A})}$$

$$\boxed{\nabla\frac{1}{|\vec{r}-\vec{r}\,'|} = -\nabla'\frac{1}{|\vec{r}-\vec{r}\,'|}}$$

$$\vec{F}(\vec{r}) = \int d\tau'\,\vec{F}(\vec{r}\,')\,\delta_D^{(3)}(\vec{r}-\vec{r}\,')$$

$$= -\frac{1}{4\pi}\int d\tau'\,\vec{F}(\vec{r}\,')\,\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|} = -\frac{1}{4\pi}\nabla^2\int d\tau'\frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= -\frac{1}{4\pi}\left[\nabla\left(\nabla\cdot\int d\tau'\frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) - \nabla\times\left(\nabla\times\int d\tau'\frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

$$= -\frac{1}{4\pi}\left[\nabla\int d\tau'\,\vec{F}(\vec{r}\,')\cdot\nabla\frac{1}{|\vec{r}-\vec{r}\,'|} - \nabla\times\left(\int d\tau'\left(\nabla\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\times\vec{F}(\vec{r}\,')\right)\right]$$

$$\cancel{\int d\tau\,\vec{A}\cdot\nabla} = -\frac{1}{4\pi}\left[-\nabla\int d\tau'\,\vec{F}(\vec{r}\,')\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}\,'|} + \nabla\times\left(\int d\tau'\left(\nabla'\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\times\vec{F}(\vec{r}\,')\right)\right]$$

$$\cancel{\int d\tau'\,\vec{A}\cdot\nabla' f = \int d\tau'\left[\nabla\cdot\left(f(\vec{r}\,')\vec{A}(\vec{r}\,')\right) - f(\vec{r}\,')\nabla\cdot\vec{A}(\vec{r}\,')\right]}$$

use: $\int d\tau\,\vec{A}\cdot\nabla f = \int d\tau\left[\nabla\cdot(f\vec{A}) - f(\nabla\cdot\vec{A})\right] = \oint f\vec{A}\cdot d\vec{a} - \int d\tau\, f(\nabla\cdot\vec{A})$

$\int d\tau\,(\nabla f)\times\vec{A} = \int d\tau\left[\nabla\times(f\vec{A}) - f(\nabla\times\vec{A})\right] = -\oint f\vec{A}\times d\vec{a} - \int d\tau\, f(\nabla\times\vec{A})$

$$\vec{F}(\vec{r}) = -\frac{1}{4\pi}\Big[+\nabla\Big(\int d\tau' \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|} - \overbrace{\oint \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot d\vec{a}'}\Big)$$

$$- \nabla\times\Big(\int d\tau' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \underbrace{\oint \frac{\vec{F}(\vec{r}')\times d\vec{a}'}{|\vec{r}-\vec{r}'|}}_{0}\Big)\Big]$$

$$= -\nabla\underbrace{\Big(+\frac{1}{4\pi}\int d\tau' \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|}\Big)}_{U(\vec{r})} + \nabla\times\underbrace{\Big(\frac{1}{4\pi}\int d\tau' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}\Big)}_{\vec{W}(\vec{r})}$$

$$\boxed{U(\vec{r}) = \frac{1}{4\pi}\int d\tau' \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|}, \quad \vec{W}(\vec{r}) = \frac{1}{4\pi}\int d\tau' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

---

* Note a corollary to Green's theorem

$$\int d\tau\, (\nabla\times\vec{A})\cdot\vec{C} = \int d\tau \Big[\nabla\cdot(\vec{A}\times\vec{C}) - \vec{A}\cdot\overbrace{(\nabla\times\vec{C})}^{0}\Big]$$

$$= \oint(\vec{A}\times\vec{C})\cdot d\vec{a} = -\oint(\vec{A}\times d\vec{a})\cdot\vec{C}$$

Since $\vec{C}$ is any constant vector

$$\boxed{\int d\tau(\nabla\times\vec{A}) = -\oint \vec{A}\times d\vec{a}}$$

integral by parts

Familiar example with functions of 1 variable

$$\int_a^b g\,\frac{df}{dx}\,dx = \int_a^b \left[\frac{d}{dx}(gf) - f\,\frac{dg}{dx}\right] dx$$

$$= \Big(g(b)f(b) - g(a)f(a)\Big) - \int_a^b f\,\frac{dg}{dx}\,dx$$

Generalize to scalar and vector fields

$$\int_a^b g(\nabla f)\cdot d\vec{l} = \int_a^b \left[\nabla(gf) - f\,\nabla g\right]\cdot d\vec{l}$$

$$= (gf)\Big|_a^b - \int_a^b f(\nabla g)\cdot d\vec{l}$$ (line integral of grad.)

$$\int_S f(\nabla\times\vec{A})\cdot d\vec{a} = \int_S \left[\nabla\times(f\vec{A}) + \vec{A}\times(\nabla f)\right]\cdot d\vec{a}$$

$$= \oint_P f\vec{A}\cdot d\vec{l} + \int_S \left[\vec{A}\times(\nabla f)\right]\cdot d\vec{a}$$ (surface integral of curl)

$$\int_V \vec{A}\cdot(\nabla f)\,d\tau = \int_V \left[\nabla\cdot(f\vec{A}) - f(\nabla\cdot\vec{A})\right] d\tau$$ (volume integral of divergence)

$$= \oint_S f\vec{A}\cdot d\vec{a} - \int_V f(\nabla\cdot\vec{A})\,d\tau$$