Recall

$\hat{L}^2 |lm\rangle \qquad |lm\rangle \leftrightarrow Y_{lm}(\theta,\varphi)$

$= l(l+1)\hbar^2 |lm\rangle$

$\hat{L}^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle$

$\hat{L}^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle$

$\hat{L}_z |lm\rangle = m\hbar |lm\rangle$

Consider a free particle of mass μ

$\hat{T}$ = kinetic energy $= \frac{p^2}{2m} = -\frac{\hbar^2}{2\mu}\nabla^2$

$p = -i\hbar\nabla$

$p^2 = -\hbar^2\nabla^2$

$$\hat{T} = -\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\hat{T} = -\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\hat{T} = -\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$= -\frac{\hbar^2}{2\mu}\Big[\underbrace{r^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)}_{\text{radial motion}} - \underbrace{\frac{\hat{L}^2}{\hbar^2 r^2}}_{\text{angular motion}}\Big]$$

Consider a particle constrained to be on the surface of a sphere of radius a (i.e. r=a)

$$\hat{T} = \frac{\hat{L}^2}{2\mu a^2}$$

$$\hat{T} = \frac{\hat{L}^2}{2\mu a^2} = \frac{\hat{L}^2}{2I}$$ ↙ moment of inertia

$\hat{V}$ indep of r

$$\hat{H} = \frac{\hat{L}^2}{2I} + \hat{V}(\theta,\varphi)$$

$$\left[\frac{L^2}{2I} + \hat{V}(\theta,\varphi)\right]\psi(\theta,\varphi) = E\psi(\theta,\varphi)$$

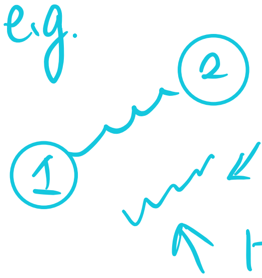
Consider $V=0$ (Rigid Rotor)

$$\hat{H} = \frac{\hat{L}^2}{2I} =$$

eigenfunctions are $Y_{lm}(\theta,\varphi)$

$$E_l = \frac{\hbar^2 l(l+1)}{2I} \quad l = 0, 1, 2, \ldots$$

e.g.



Assume this vibrating motion is small

Harmonic Oscillator

- Can describe rotational motion in diatomic molecules

Scalar: E (continuous, discrete)

Vector: $\vec{P}$, $\vec{X}$ (continuous, discrete) $[\hat{X}, \hat{P}] = i\hbar$

$\left.\vec{L} = \vec{r} \times \vec{p}\right\} \rightarrow$ quantized vector L^2, $\hat{L}$,

Generalized Angular Momentum

- $\vec{J}$ $\quad$ $\vec{J}_x, \vec{J}_y, \vec{J}_z$
- $[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$

  $[\hat{J}^2, \hat{J}_z] = 0$

- $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$

  $\hat{J}_- = \hat{J}_x - i\hat{J}_y$

  $\hat{J}_+^\dagger = \hat{J}_- \quad \hat{J}_-^\dagger = \hat{J}_+$

$\hat{J}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$

$J_z |jm\rangle = m\hbar |jm\rangle$

$j(j+1) > m^2$

(length of the vector)^2 $\quad$ (z projection of the vector)^2

Assume we have a maximum / minimum of m

$\rightarrow m_T$ top of ladder

$\rightarrow m_B$ bottom

$$\hat{J}_+ |jm\rangle = [j(j+1) - m(m+1)] |jm\rangle$$

$$\hat{J}_- |jm\rangle = [j(j+1) - m(m-1) \, |j\, m-1$$

$$\hat{J}_\pm \hat{J}_\mp = \hat{J}^2 - \hat{J}_z \pm \hbar \hat{J}_z$$

$$\hat{J}_+ |jm_T\rangle = 0$$

$$\hat{J}_- (\hat{J}_+ |jm_T\rangle) = (\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z) |jm_T\rangle$$

$$= [j(j+1) - m_T^2 - m_T] \hbar^2 |jm_T\rangle$$

$$\longrightarrow j(j+1) = m_T^2 + m_T$$

Similarily

$$\hat{J}_- |jm_B\rangle = 0$$

$$\hat{J}_+ (\hat{J}_- |jm_B\rangle) = (\hat{J}^2 - \hat{J}_B^2 + \hbar \hat{J}_B) |jm_B\rangle$$

$$= [j(j+1) - m_B^2 + m_B] \hbar^2 |jm_B\rangle$$

$$j(j+1) = m_B^2 - m_B$$

$$m_T^2 + m_T = m_B^2 - m_B$$

Either $m_T = -m_B$

or $m_T = m_B - 1$ ✗ Contradicts the idea that $m_T - T_B = n \leftarrow$ integer

Let $m_T = +j$

$m_B = -j$

$m_T - m_B = 2j \rightarrow$ positive integer

$\rightarrow j = 0, \frac{1}{2}, 1, \frac{3}{2}$!!!!!!!!

$j = 0 \quad J^2 = 0$

$m = 0 \quad J_z = 0$

$j = \frac{1}{2}$

$m = -\frac{1}{2}, \frac{1}{2} \quad J^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$j = 1$

$m = +1, 0, -1$

$$J^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} (j=0) & & \\ & \left(j = \frac{1}{2} \right) & \\ & & (j = 1) \end{pmatrix}$$