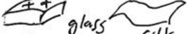
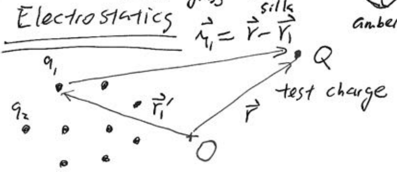


Electrostatics

Source and test charges are either "stationary or "slowly" moving



~~stationary~~ source charges $q_1, q_2, \cdots, q_N$

length scale of the system $\sim \ell$

time scale of the system $\sim \tau$

we need $\boxed{\ell \ll \tau c}$

size of a room: $\ell \sim 10\,m$

time scale of AC current: $\tau \sim 0.01\,s$

(rapid processes) time scale of experiments: $\tau \sim 1\,\mu s$

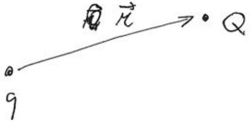
$$\tau c = 1\,\mu s \cdot 3\times10^{8}\,m/s = 300\,m \gg \ell$$

$$\tau c = 0.01\,s \cdot 3\times10^{8}\,m/s = \mathcal{O}(1000\,km) \gg \ell$$

(French military engineer & physicist)

Coulomb's Law

(q, Q can be positive or negative)



$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\imath^2}\hat{\imath}, \quad \hat{\imath} \equiv \frac{\vec{\imath}}{|\vec{\imath}|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\imath^3}\vec{\imath}$$

"inverse square" law

SI units: (kg, m, sec)

distance ..... meters (m)

charge ..... Coulomb (C)

Force ..... Newton (N)

$1\,N = (1\,kg)\cdot(1\,m s^{-2})$

$1\,(dyne) = (1\,g)\cdot(1\,cm s^{-2})$

$1\,N = 10^5\,dyne$

$$\boxed{\varepsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N\cdot m^2}}$$ ...... permittivity of free space

Gaussian units: (g, cm, sec)

$$\vec{F} = \frac{qQ}{\imath^2}\hat{\imath}$$

charge .... electrostatic units (esu)

$1\,esu = 1(dyne)^{1/2}\cdot cm$

Quantization of electric charge

elemetary charge

charge of an electron $-e = -1.6 \times 10^{-19}$ C

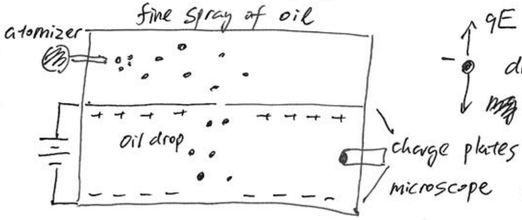
charge of proton $e = 1.6 \times 10^{-19}$ C

Millikan ~~Experiment~~ Oil Drop Experiment

Harvey Fletcher
Robert Millikan 1909
(Nobel Prize 1923)

J. J. Thomson discovered e/m_e

charging by X-rays



$\uparrow qE$

droplet suspend in air

$\downarrow$ ~~mg~~ F_G

$E = V/d$

$$qE = W = \frac{4\pi}{3} r^3 (\rho - \rho_{air}) g$$

electric field turned off $\frac{4\pi}{3} r^3 (\rho - \rho_{air}) g = 6\pi r \eta v_1$ ← terminal velocity

η ↑ air viscocity

No electric field

spherical droplet $m = \frac{4\pi}{3} r^3 \rho$

air resistance $\uparrow$ $6\pi r \eta v_1$

$\uparrow \frac{4\pi}{3} r^3 \rho_{air}\, g$ (buoyancy)

$\downarrow v_1$

$\downarrow \frac{4\pi}{3} r^3 \rho g$ … gravity

$$\frac{4\pi}{3} r^3 (\rho - \rho_{air}) g = 6\pi r \eta v_1$$

With electric field

$qE \uparrow$

$\uparrow \frac{4\pi}{3} r^3 \rho_{air}\, g$

$\uparrow v_2$

$6\pi r \eta v_2 \downarrow$

$\downarrow \frac{4\pi}{3} r^3 \rho g$

$$\frac{4\pi}{3} r^3 (\rho - \rho_{air}) g + 6\pi r \eta v_2 = qE$$

Find that q is always integer multiples of -1.6×10^{-19} C

~~charges~~ oil droplets carry electrons

Principle of superposition

N charges $a = 1, 2, 3, \cdots, N$

$$\vec{F} = \sum_{a=1}^{N} \vec{F}_a = \sum_{a=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_a Q}{|\vec{r} - \vec{r}_a|^3} (\vec{r} - \vec{r}_a)$$

$$= Q \underbrace{\sum_{a=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_a}{|\vec{r} - \vec{r}_a|^3}}_{\text{electric field } \vec{E} \text{ at } \vec{r}}$$

location of Q

electric field $\vec{E}$ at ~~field point~~ $\vec{r}$

$$\boxed{\vec{F} = Q\vec{E}}$$

Continuous charge distribution

quantization of charges?

Point charge q • (0d)

line charge (1d)

$dq = \lambda \, dl$

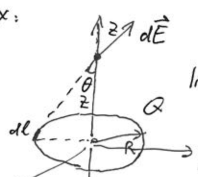
(2d) Surface charge

$dq = \sigma \, da$

(3d) volume charge

$dq = \rho \, d\tau$

Ex:



line charge $\lambda = \frac{Q}{2\pi R}$

$$dq = \lambda \, dl = \lambda R \, d\phi = \frac{Q}{2\pi} d\phi$$

$$dE_z = \cos\theta \, dE = \cos\theta \frac{1}{4\pi\varepsilon_0} \frac{\lambda \, dl}{z^2 + R^2}$$

$$= \frac{Q\cos\theta}{4\pi\varepsilon_0 \, 2\pi R} \frac{R \, d\phi}{z^2 + R^2}$$

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{\cos\theta}{2\pi R} \frac{R}{z^2 + R^2} \left(\int_0^{2\pi} d\phi \right)$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{1}{z^2 + R^2} \frac{z}{\sqrt{z^2 + R^2}} = \frac{Q}{4\pi\varepsilon_0} \frac{z}{(z^2 + R^2)^{3/2}}$$

if $z \gg R$, $E \to \frac{Q}{4\pi\varepsilon_0 z^2}$ equivalent to a point charge Q

~~Continuous charge distribution~~

Gauss's Law

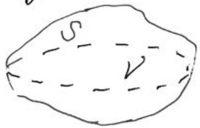
$$\nabla\cdot\left(\frac{1}{|\vec r-\vec r_a|^3}(\vec r-\vec r_a)\right) = \cancel{0}\; 4\pi\,\delta_D^{(3)}(\vec r-\vec r_a) \quad \cancel{\text{except } \vec r=\vec r_a}$$

$$\nabla\cdot\vec E = \nabla\cdot\left[\sum_{a=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_a}{|\vec r-\vec r_a|^3}(\vec r-\vec r_a)\right]$$

$$= \sum_{a=1}^{N}\frac{q_a}{4\pi\varepsilon_0}\nabla\cdot\left(\frac{\vec r-\vec r_a}{|\vec r-\vec r_a|^3}\right) = \sum_{a=1}^{N}\frac{q_a}{4\pi\varepsilon_0}(+4\pi)\,\delta_D^{(3)}(\vec r-\vec r_a)$$

Use Green's Theorem

$$\oint_S \vec E\cdot d\vec a = \int_V(\nabla\cdot\vec E)\,d\tau = +\frac{1}{\varepsilon_0}\underbrace{\left(\sum q_a\right)}^{\text{enclosed charge}}$$



$$\boxed{\oint_S \vec E\cdot d\vec a = \frac{Q_{enc}}{\varepsilon_0}}$$

$$\vec E(\vec r) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\,\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|^3}\,\rho(\vec r\,')$$

$$\nabla\cdot\vec E(\vec r) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\,\nabla\cdot\left(\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|^3}\right)\rho(\vec r\,') = \frac{1}{\varepsilon_0}\int d\tau'\,\rho(\vec r\,')\,\delta_D^{(3)}(\vec r-\vec r\,')$$

$$= \frac{1}{\varepsilon_0}\rho(\vec r)$$

$$\boxed{\nabla\cdot\vec E = \frac{\rho}{\varepsilon_0}}$$

Curl of $\vec E$

$$\vec E(\vec r) = \frac{-1}{4\pi\varepsilon_0}\int d\tau'\,\nabla\left(\frac{\rho(\vec r\,')}{|\vec r-\vec r\,'|}\right)$$

$$\vec\nabla\times\vec E(\vec r) = -\frac{1}{4\pi\varepsilon_0}\int d\tau'\,\vec\nabla\times\nabla\left(\frac{\rho(\vec r\,')}{|\vec r-\vec r\,'|}\right) = 0$$

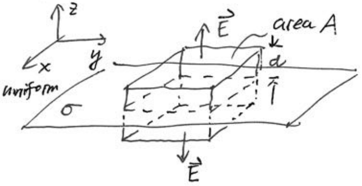
$$\boxed{\nabla\times\vec E = 0}$$

$$\oint_P \vec E\cdot d\vec l = \int_S(\nabla\times\vec E)\cdot d\vec a = 0 \quad \text{for any closed } P$$

$$\Leftrightarrow \int_a^b \vec E\cdot d\vec l \quad \text{independent of path } a\to b$$

$$\Leftrightarrow \vec E = \nabla U$$

Application of Gauss's Law



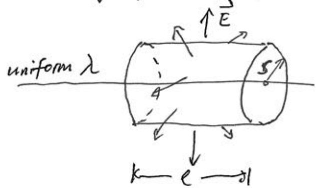
infinitely large ~~surface~~ planar charge

$$2EA = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

independent of distance!

infinitely long line charge

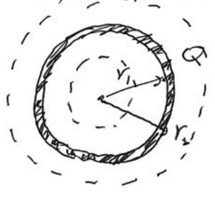


$$2\pi r \ell E(s) = \frac{\lambda \ell}{\varepsilon_0}$$

$$\boxed{E(s) = \frac{\lambda}{2\pi\varepsilon_0 s}}$$

E(s) scales as $\frac{1}{s}$

uniform spherical shell



if $r > r_2$ $\quad 4\pi r^2 E(r) = \frac{Q}{\varepsilon_0}$

$$\Rightarrow E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

same as point ~~mass~~ charge

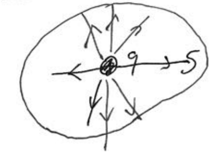
if $r < r_1$ $\quad 4\pi r^2 E(r) = 0$

$$\Rightarrow E(r) = 0$$

no E field inside!

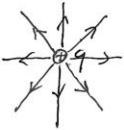
Electric field lines

$\oint_S \vec{E} \cdot d\vec{a}$ Gauss's law

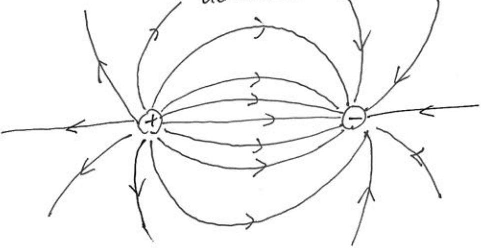


$$(\text{flux through } S) = (\#\text{ of E lines coming out})$$

$$\#\text{ of E lines} = \frac{q}{\varepsilon_0}$$



attraction



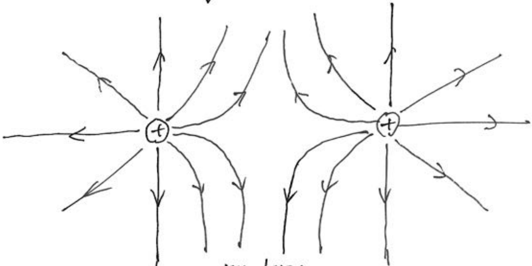
E-lines do not intersect with each other

~~① Start from positive charges, end a~~

- positive charge $\longrightarrow$ negative charges
- positive charge $\longrightarrow$ infinity
- infinity $\longrightarrow$ negative charge

* E field strength $|\vec{E}| \propto$ number density of E lines

*



repulsion

Curl of $\vec{E}$

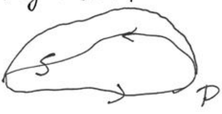
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')$$

$$= \frac{-1}{4\pi\varepsilon_0}\int d\tau' \nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\rho(\vec{r}')$$

$$= -\nabla\left(\frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$$

Hence $\boxed{\nabla \times \vec{E}(\vec{r}) = 0}$ Curl-free field

Any closed path



$$\oint_P \vec{E}\cdot d\vec{\ell} = \int_S (\nabla\times\vec{E})\cdot d\vec{a} = 0$$

$\Leftrightarrow$ $\int_a^b \vec{E}\cdot d\vec{\ell}$ is path independent

$\Leftrightarrow$ $\vec{E}$ is gradient of some scalar field!

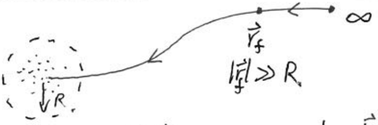
Define electric potential $\boxed{\vec{E} = -\nabla V}$

(minus sign!)

$$\boxed{V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{\ell}}$$

* V decreases along E lines

Localised ~~system~~ distribution of charges



caveat, infinitely long line charge, infinitely large surface charge, etc.

$$\vec{E}(\vec{r}_f) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\vec{r}_f-\vec{r}'}{|\vec{r}_f-\vec{r}'|^3}\rho(\vec{r}') \simeq \frac{1}{4\pi\varepsilon_0}\frac{\vec{r}_f}{|\vec{r}_f|^3}\int d\tau' \rho(\vec{r}') \propto \frac{1}{r_f^2}$$

$$\int_\infty^{\vec{r}_f} \vec{E}\cdot d\vec{\ell} \propto \int_\infty^{\vec{r}_f} \frac{dl}{r^2} \quad \text{finite!}$$

$$\int_{\vec{r}_f}^{\vec{r}} \vec{E}\cdot d\vec{\ell} \quad \text{finite!}$$

so we can pick $V(\infty) = 0$

$$\boxed{V(\vec{r}) = -\int_\infty^{\vec{r}} \vec{E}\cdot d\vec{\ell}}$$

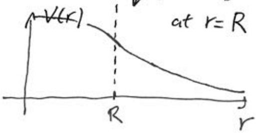
Electric potential: principle of superposition

$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|} \quad \text{or} \quad V(\vec r) = \sum_{a=1}^{N} \frac{q_a}{4\pi\varepsilon_0}\frac{1}{|\vec r - \vec r_a|}$$

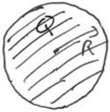
$$\cancel{\nabla^2 V = \nabla\cdot(\nabla V) = -\nabla\cdot\vec E = -\frac{\rho}{\varepsilon_0}}$$

$$\cancel{\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}} \quad \text{~~Poisson's Equation~~}$$

V, $\frac{dV}{dr}$ both continuous at $r=R$



Ex: Uniformly charged ball



$$\vec E = E(r)\,\hat r$$

$$E(r) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}, & r>R \\[2ex] \dfrac{1}{4\pi\varepsilon_0} Q\left(\dfrac{r^3}{R^3}\right)\dfrac{1}{r^2} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Qr}{R^3}, & r<R \end{cases}$$

if $r>R$, $\quad V(r) = -\int_\infty^r \vec E\cdot d\vec\ell = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0}\frac{dr'}{r'^2} = \frac{+Q}{4\pi\varepsilon_0}\left(\frac{1}{r'}\right)\Big|_\infty^r = \frac{Q}{4\pi\varepsilon_0 r}$

if $r<R$, $\quad V(r) \overset{V(R)}{=} -\int_R^r \vec E\cdot d\vec\ell = V(R) - \int_R^r \frac{Q}{4\pi\varepsilon_0 R^3} r'\,dr'$

$$= \frac{Q}{4\pi\varepsilon_0 R} + \frac{Q}{4\pi\varepsilon_0 R^3}\frac{1}{2}(R^2 - r^2) = \frac{Q}{4\pi\varepsilon_0 R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right)$$

Laplacian of V

$$\nabla^2 V = \nabla\cdot(\nabla V) = -\nabla\cdot\vec{E} = -\frac{\rho}{\varepsilon_0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$ Poisson's equation

in region of vacuum (no charges)

$$\boxed{\nabla^2 V = 0}$$ ...... Laplace's equation

~~boundary conditions at~~

continuity of electric potential and continuous

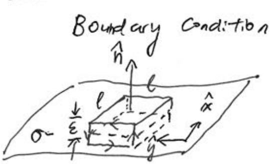
* if only volume charge, $\vec{E}$ is finite as $|\vec{r}-\vec{r}'| \to 0$

($\rho(\vec{r}')$ finite) $\quad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')$, $\quad |\vec{E}| \sim \frac{1}{|\vec{r}-\vec{r}'|^2}$, $\quad d\tau \sim |\vec{r}-\vec{r}'|^3$

$$\partial_j E_i = \frac{1}{4\pi\varepsilon_0}\int \underbrace{d\tau'}_{\sim \mathscr{r}^3} \frac{1}{\mathscr{r}^3}\underbrace{\left(\delta_{ij} - 3\frac{\mathscr{r}_i \mathscr{r}_j}{\mathscr{r}^2}\right)}_{\text{orthogonal to } \delta_{ij}} \rho(\vec{r}')$$

* since $dV = -\vec{E}\cdot d\vec{l}$, V is continuous

Boundary Condition on both sides of ~~the~~ surface charge



$$E_n = \vec{E}\cdot\hat{n} \qquad \frac{\partial V}{\partial n} \triangleq (\nabla V)\cdot\hat{n} \text{ normal deriv.}$$

$$(E_n)_+\ell^2 - (E_n)_-\ell^2 + \overbrace{(\text{sides})}^{\propto \varepsilon\ell} = \frac{\sigma\ell^2}{\varepsilon_0}$$

so $$\boxed{(E_n)_+ - (E_n)_- = \frac{\sigma}{\varepsilon_0}, \quad \left(\frac{\partial V}{\partial n}\right)_+ - \left(\frac{\partial V}{\partial n}\right)_- = -\frac{\sigma}{\varepsilon_0}}$$

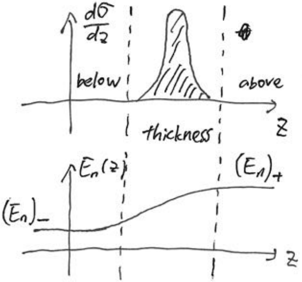
$$(E_y)_+\ell - (E_y)_-\ell + \overbrace{(\text{sides})}^{\propto \varepsilon} = 0 \Rightarrow \boxed{(\vec{E}_\parallel)_+ = (\vec{E}_\parallel)_-}$$

$$\boxed{\vec{E}_+ - \vec{E}_- = \frac{\sigma}{\varepsilon_0}\hat{n}}$$ what about line charge, point charge?

Electric force acting on ~~the~~ surface charge

$F_n = \sigma E_n$?

which E_n ?



$$F_n = \int_{z_-}^{z_+} dz \, \frac{d\sigma}{dz}(z)\, E_n(z)$$

Gauss's law

$$= \varepsilon_0 \int_{z_1}^{z_+} dz \, \frac{dE_n(z)}{dz} E_n(z)$$

$$= \frac{\varepsilon_0}{2} E_n^2(z) \Big|_{z_1}^{z_2} = \frac{\varepsilon_0}{2} \overbrace{\left((E_n)_+ - (E_n)_-\right)}^{\frac{\sigma}{\varepsilon_0}} \cdot \left((E_n)_+ + (E_n)_-\right)$$

$$F_n = \frac{\sigma}{2}\left((E_n)_+ + (E_n)_-\right)$$

Work and Energy in Electrostatics

External electric field $\vec{E}(\vec{r})$

test ~~particle~~ charge ~~q~~ Q

"slowly" move the charge

work done by electrostatic force

$$W = \int_a^b \vec{F}\cdot d\vec{\ell} = -\int_a^b (Q\vec{E})\cdot d\vec{\ell} = +Q\left(V(b) - V(a)\right)$$

or
$$\boxed{V(b) - V(a) = \frac{W}{Q}}$$

move from "infinity" to $\vec{r}$: $\quad W = Q\left(V(\vec{r}) - V(\infty)\right) = \underbrace{QV(\vec{r})}_{\text{electrostatic energy}}$

Assemble N charges: $q_1, q_2, q_3, \dots, q_N$
to positions: $\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_N$

* First charge no work done $W_1 = 0$

* Second charge $\quad W_2 = \cancel{q_2}\, q_2 \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{r}_2 - \vec{r}_1|}$

* 3rd charge $\quad W_3 = q_3 \cancel{q_3}\left(\frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{|\vec{r}_3 - \vec{r}_2|}\right)$

⋮

* Nth charge $\quad W_N = q_N \cancel{q_N}\frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N-1}\frac{q_j}{|\vec{r}_N - \vec{r}_j|}$

Total work done

$$W_1 + W_2 + W_3 + \cdots + W_N = \underbrace{\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{i-1}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}_{\text{sum over all pairs!}}$$

$$= \underline{\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_i\sum_{j\neq i}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}$$ ← Electrostatic Energy of the system

$$= \frac{1}{2}\sum_{i=1}^{N} q_i \underbrace{\left(\sum_{j\neq i}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{|\vec{r}_i - \vec{r}_j|}\right)}_{V(\vec{r}_i)\ \text{(excluding self field!)}}$$

Energy of Continuous charge distribution

$$\boxed{W = \frac{1}{2}\int d\tau\, \rho(\vec{r})\, V(\vec{r})}$$

self-energy ?? Consider a uniformly charged ball of density ρ and radius R

electric potential

$$V(r) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{4\pi}{3}R^3\rho\right)\dfrac{1}{r} = \dfrac{\rho R^3}{3\varepsilon_0}\dfrac{1}{r}, & r > R \\[2ex] \dfrac{\rho R^2}{3\varepsilon_0}\left(\dfrac{3}{2} - \dfrac{1}{2}\dfrac{r^2}{R^2}\right), & r < R \end{cases}$$

self energy $\quad \mathcal{E} = \dfrac{\rho}{2}(4\pi)\displaystyle\int_0^R r^2 dr\, \dfrac{\rho R^2}{3\varepsilon_0}\left(\dfrac{3}{2} - \dfrac{1}{2}\dfrac{r^2}{R^2}\right)$

$$= \frac{\rho^2 R^2}{6\varepsilon_0}(4\pi)\underbrace{\int_0^R r^2 dr\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right)}_{\left(\frac{3}{2}\frac{1}{3} - \frac{1}{2}\cdot\frac{1}{5}\right)R^3 = \frac{2}{5}R^3} = \frac{4\pi\rho^2 R^5}{15\varepsilon_0} \xrightarrow[\substack{R\to 0 \\ \text{fix } \rho}]{} 0$$

Total charge $\quad q = \dfrac{4\pi}{3}R^3\rho$

Do this another way!

$$\mathcal{E} = \frac{4}{15\varepsilon_0}R^5\left(\frac{3q}{4\pi R^3}\right)^2 = \frac{3q^2}{20\pi\varepsilon_0 R} \xrightarrow[\substack{R\to 0 \\ \text{fix } q}]{} +\infty$$

~~diverge as 1/R because~~

$$W = \frac{1}{2}\int d\tau\, \rho V = \frac{\varepsilon_0}{2}\int d\tau\, (\nabla\cdot\vec{E})V = \frac{\varepsilon_0}{2}\int d\tau\left[\nabla\cdot(\vec{E}V) - \vec{E}\cdot\nabla\vec{V}\right]$$

$$= \frac{\varepsilon_0}{2}\left[\underbrace{\oint_S V\vec{E}\cdot d\vec{a}}_{} + \int d\tau\, \vec{E}\cdot\vec{E}\right]$$

$=0$ at infinity given that $|\vec{E}| \propto \frac{1}{r^2}$, $V \propto \frac{1}{r}$, $|d\vec{a}| \propto r^2$

Therefore $\quad \boxed{W = \dfrac{\varepsilon_0}{2}\displaystyle\int d\tau\, \vec{E}^2}$ (all space)

Energy density of static electric field $\frac{1}{2}\varepsilon_0\vec{E}^2$

Two points of view:

① Electrostatic energy stored between "pairs" of charges

② Electrostatic energy stored in electric field $\vec{E}$

$\frac{\varepsilon_0}{2}\int d\tau\, \vec{E}^2$ gives the total energy, but is divergent if there are point charges $\int d\tau\, \vec{E}^2 \sim \int r^2 dr \left(\frac{1}{r^2}\right)^2 \sim \int \frac{dr}{r^2}$

puzzle: a pair of positive and negative charges

$$W = \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{r} = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{r} < 0$$

$$W = \frac{\varepsilon_0}{2}\int d\tau\, \vec{E}^2 > 0$$

No superposition principle!

$$W_{tot} \neq W_A + W_B$$

A B

Two charge systems A, B

$$W_{tot} = \frac{\varepsilon_0}{2}\int d\tau\, \vec{E}^2 = \frac{\varepsilon_0}{2}\int d\tau \left(\vec{E}_A + \vec{E}_B\right)^2$$

$$= \underbrace{\frac{\varepsilon_0}{2}\int d\tau\, \vec{E}_A^2}_{W_A} + \underbrace{\frac{\varepsilon_0}{2}\int d\tau\, \vec{E}_B^2}_{W_B} + \varepsilon_0\int d\tau\, \vec{E}_A\cdot\vec{E}_B$$

dQ

$$Q = \frac{4\pi}{3} r^3 \rho$$

add charge to ~~cha~~surface layer by layer

$$dQ = 4\pi r^2 \rho \, dr$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{3}\rho r^2 = \frac{\rho}{3\varepsilon_0} r^2$$

$$dU = V\,dQ = \frac{\rho}{3\varepsilon_0} r^2 \, 4\pi\rho r^2 dr = \frac{4\pi\rho^2}{3\varepsilon_0} r^4 dr$$

$$U = \int_0^R dr \frac{4\pi\rho^2}{3\varepsilon_0} r^4 dr = \frac{4\pi\rho^2}{3\varepsilon_0}\frac{R^5}{5} = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{Q^2}{R}$$

Same answer !!

Classic electron radius

$-e$

r_e

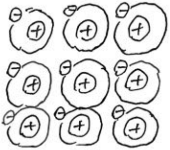
electrostatic energy $U = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{e^2}{r_e} \sim \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_e} > 0$

According to special relativity $U = m_e c^2$ (speed of light; electron mass)

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_e} \simeq m_e c^2 \Rightarrow r_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e c^2} \simeq 2.8 \times 10^{-15}\,\text{m}$$

(several times larger than the radius of the proton)

Conductors



Insulators: glass, rubber, plastic

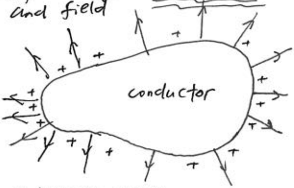
Conductors: Copper, silver, iron, graphene, plasma

insulators and conductors can turn into each other under the right physical conditions.

Examples: air? vacuum?

(ideal) Property of Conductors

Surface charge and field $\sigma = \varepsilon_0 E$



① $\vec{E} = 0$ inside

② $\rho = 0$ inside ($\nabla \cdot \vec{E} = \rho/\varepsilon_0$)

③ Any net charge resides on surface

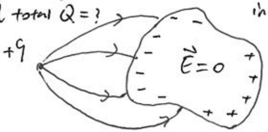
④ Equipotential $V = \text{const.}$

⑤ E field normal to surface

⑥ minimizes electrostatic energy, given total charge q_{tot} (conserved)

2 ways to specify:
$\begin{cases} V = ? \\ \text{total } Q = ? \end{cases}$

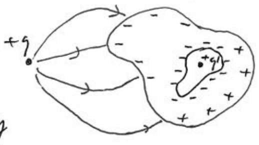
induced charges



hollow cavity

metallic cage?

electrostatic screening

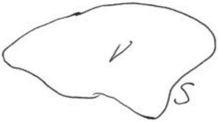


uncharged ball with a cavity where there is a charge

uniqueness of solutions

field outside of the conducting ball?

Uniqueness of solution to electrostatic problem



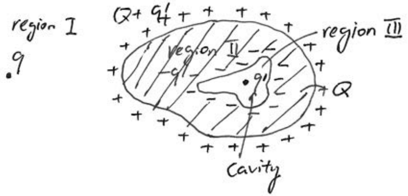
Need to satisfy:

① In V, Poisson's equation holds for V or Gauss's equation holds for $\vec{E}$

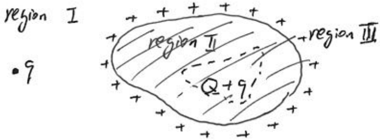
② Boundary conditions at S are met

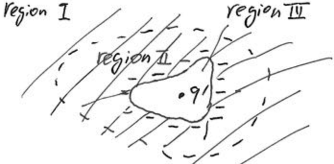
(uniqueness theorems for solutions to ~~the~~ Poisson's equation)

* Consider electrostatic problem involving a conductor



Consider two other problems





① region I: $\nabla\cdot(\vec{E}_q + \vec{E}_{out}) = \rho_{I}/\epsilon_0$

② region II: $\vec{E}_q + \vec{E}_{out} = 0$

③ region III: $\vec{E}_q + \vec{E}_{out} = 0$

① region I: $\vec{E}_{in} + \vec{E}_{q'} = 0$

② region II: $\vec{E}_{in} + \vec{E}_{q'} = 0$

③ region III: $\nabla\cdot(\vec{E}_{in} + \vec{E}_{q'}) = \rho_{III}/\epsilon_0$

Combine two solutions:

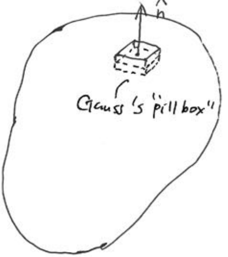
① region I: $\nabla\cdot(\vec{E}_q + \vec{E}_{out} + \vec{E}_{in} + \vec{E}_{q'}) = \rho_I/\epsilon_0$

② region II: $\vec{E}_q + \vec{E}_{out} + \vec{E}_{in} + \vec{E}_{q'} = 0$

③ region III: $\nabla\cdot(\vec{E}_q + \vec{E}_{out} + \vec{E}_{in} + \vec{E}_{q'}) = \rho_{III}/\epsilon_0$

Exactly what we need for the original problem!

Surface charge and electrostatic force acting on conductor



$$(E_n)_+ - (E_n)_- = \frac{\sigma}{\varepsilon_0} \quad \text{(use Gauss's Law)}$$

$(E_n)_- = 0$ (interior of conductor)

Therefore $\boxed{\vec{E}_+ = \frac{\sigma}{\varepsilon_0}\hat{n}}$

Normal gradient $\boxed{\left(\frac{\partial V}{\partial n}\right)_+ = -\frac{\sigma}{\varepsilon_0}}$

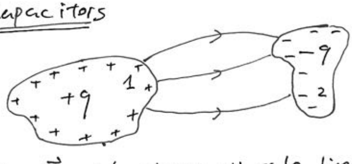
force per unit area $\vec{f} = \frac{1}{2}\sigma\vec{E}_+ = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$

$$\begin{cases} \vec{E}_+ = \vec{E}_{ext} + \vec{E}_{self}, & \text{where } \vec{E}_{self} = \frac{\sigma}{2\varepsilon_0}\hat{n} \\ \vec{E}_- = \vec{E}_{ext} - \vec{E}_{self} & \Rightarrow \vec{E}_{ext} = \vec{E}_{self} \end{cases}$$

Therefore $\underline{\vec{f} = \sigma\vec{E}_{ext} = \frac{1}{2}\sigma\vec{E}_+}$

Discussion: Stronger fields near sharp end.

Capacitors



voltage $V = \int_1^2 \vec{E}\cdot d\vec{l}$

~~capacity~~ capacitance $\boxed{C \triangleq \frac{q}{V}}$

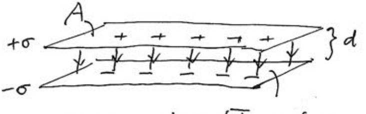
σ, $\vec{E}$, V should all scale linearly with q

if 2 is removed to infinity, then $V = \int_1^\infty \vec{E}\cdot d\vec{l}$

capacitance can still be defined.

SI unit for capacitance: farad $\quad 1\,F = \frac{1\,C}{1\,V}$

Examples



$$\sigma = \frac{q}{A}, \quad E = \frac{\sigma}{\varepsilon_0}$$

$$V = Ed = \frac{\sigma d}{\varepsilon_0}$$

$$C = \frac{q}{V} = q\frac{\varepsilon_0}{\sigma d} = q\frac{\varepsilon_0 A}{q d} = \frac{\varepsilon_0 A}{d}$$

assume $d \ll \sqrt{A}$ uniform E-field in between two plates

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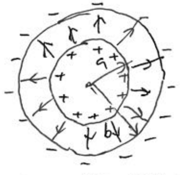


concentric metal spherical shells

for $a < r < b$

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{radial } E \text{ field}$$

$$V = \int_a^b E_r\,dr = \int_a^b \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{q}{V} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$

Energy of Capacitor

move dq from negative conductor to positive conductor

$q \to q + dq$

$$dU = V\,dq = \frac{q}{C}dq$$

$$U = \int_0^q \frac{q'}{C}dq' = \frac{q^2}{2C} = \frac{1}{2}CV^2$$

Energy for parallel plates

$$U = Ad\frac{\varepsilon_0}{2}E^2 = \frac{Ad}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{Ad}{2}\varepsilon_0\left(\frac{q}{A\varepsilon_0}\right)^2 = \frac{q^2}{2C}$$

Attraction per unit area $f = \frac{\sigma^2}{2\varepsilon_0} = \frac{q^2}{2\varepsilon_0 A^2}$

$$Af\,\Delta d = \Delta U = \frac{q^2}{2C^2}\Delta C = \frac{q^2}{2C^2}\frac{C}{d}\Delta d \Rightarrow f = \frac{q^2}{2CdA} = \frac{q^2}{2\varepsilon_0 A^2}$$

Work done converted into electrostatic energy (stored in field)

Analogy between electrostatics & Newton's gravity

Coulomb's law $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1)$

Newton's law $\vec{F}_{12} = -G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_2 - \vec{r}_1)$, $m_1, m_2 > 0$

Electric field $\vec{F} = q\vec{E}$, $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d\tau' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} (\vec{r} - \vec{r}\,')$

gravitational acceleration

curl-free $\nabla \times \vec{E} = 0$ $\quad \vec{F} = m\vec{g}$, $\vec{g}(\vec{r}) = -G \int d\tau' \frac{\rho_m(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} (\vec{r} - \vec{r}\,')$

$\nabla \times \vec{g} = 0$

Gauss's law $\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$ or $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$\oint \vec{g} \cdot d\vec{a} = -4\pi G m_{enc}$ or $\nabla \cdot \vec{g} = -4\pi G \rho_m$

---

electric potential $V = -\int_\infty^{\vec{r}} \vec{E} \cdot d\vec{l}$, $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d\tau' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$

gravitational potential $\Phi = -\int_\infty^{\vec{r}} \vec{g} \cdot d\vec{l}$, $\Phi(\vec{r}) = -G \int d\tau' \frac{\rho_m(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} < 0$

Poisson's Equation $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

$\nabla^2 \Phi = 4\pi G \rho_m$

Electrostatic energy $U = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$

$U = \frac{1}{2} \int d\tau' \rho(\vec{r}\,') V(\vec{r}\,') = \frac{\varepsilon_0}{2} \int d\tau \, \vec{E}^2$

gravitational binding energy $U = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} G \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|} < 0$

$U = \frac{1}{2} \int d\tau \, \rho_m \Phi = -\frac{1}{8\pi G} \int d\tau \, \vec{g}^2$

Economic interpolation — Laplace's equation

"boring"

$$\nabla^2 V = 0 \quad \text{minimizes} \quad \phi = \int d\tau\, (\nabla V)^2 \quad \text{for given } V|_S$$

consider $V' = V + \delta V\,, \quad \delta V|_S = 0$

$$\phi(V') = \int_V d\tau \left[\nabla(V+\delta V)\right]^2 = \phi(V) + 2\int_V d\tau (\nabla V)\cdot(\nabla \delta V) + \underbrace{\int_V d\tau (\nabla \delta V)^2}_{>0}$$

$$\int_V d\tau\, (\nabla V)\cdot(\nabla \delta V) = \int_V d\tau \left(\nabla\cdot(\delta V \nabla V) - \delta V \nabla^2 V\right)$$

$$= \oint_S d\vec{a}\cdot(\underbrace{\delta V}_{=0} \nabla V) - \int_V d\tau\, \delta V \underbrace{\nabla^2 V}_{=0} = 0$$

Hence $\phi(V') > \phi(V)$ unless $\delta V \equiv 0$ (because $\delta V|_S = 0$)

Stable electrostatic equilibrium in vacuum?

consider a test charge Q at $\vec{r}_Q$

equilibrium requires $\nabla V(\vec{r}_Q) = 0$

Taylor expand $V(\vec{r}) = V(Q) + \underbrace{(\nabla V)_Q}_{=0}\cdot(\vec{r} - \vec{r}_Q)$

$$+ \tfrac{1}{2}(\nabla_i \nabla_j V)_Q (\vec{r} - \vec{r}_Q)_i (\vec{r} - \vec{r}_Q)_j + \cdots$$

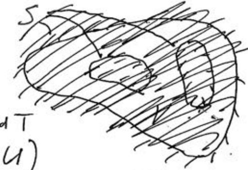
For any direction of $\vec{r} - \vec{r}_Q$, we need

$$\frac{1}{2}(\nabla_i \nabla_j V)_Q (\vec{r} - \vec{r}_Q)_i (\vec{r} - \vec{r}_Q)_j \begin{cases} > 0, & \text{if } Q > 0 \\ < 0, & \text{if } Q < 0 \end{cases}$$

This requires $\nabla_i \nabla_i V = \nabla^2 V \begin{cases} > 0, & \text{if } Q > 0 \\ < 0, & \text{if } Q < 0 \end{cases}$

However $\nabla^2 V = 0$, contradiction!

Laplace's equation; mean value theorem



Use Green's identity: two scalar functions U and T

$$\int_V d\tau \left(U\nabla^2 T - T\nabla^2 U\right) = \int_V d\tau\, \nabla\cdot\left(U\nabla T - T\nabla U\right)$$

$$= \oint_S d\vec{a}\cdot\left(U\nabla T - T\nabla U\right)$$

let $U = V(\vec{r})$, $T = \dfrac{1}{|\vec{r}-\vec{r}'|}$

$$\int_V d\tau' \Big(V(\vec{r}')\underbrace{\nabla'^2 \frac{1}{|\vec{r}-\vec{r}'|}}_{-(4\pi)\,\delta_D^{(3)}(\vec{r}-\vec{r}')} - \frac{1}{|\vec{r}-\vec{r}'|}\overbrace{\nabla'^2 V(\vec{r}')}^{=0\ \text{Laplace's eq.}}\Big)$$

$$= -(4\pi)V(\vec{r})$$

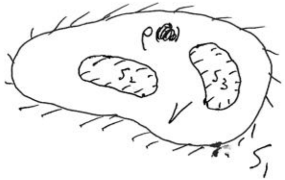
This equals $\displaystyle\oint_S d\vec{a}'\cdot\left(V(\vec{r}')\nabla'\frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'|}\nabla' V(\vec{r}')\right)$

$$= -\oint_S V(\vec{r}')\,\frac{da'\,\hat{n}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \frac{1}{R}\underbrace{\oint_S d\vec{a}'\cdot\left(-\nabla' V(\vec{r}')\right)}_{\text{no enclosed charge } q_{enc}=0}$$

$$= -\frac{1}{R^2}\oint_S V(\vec{r}')\,da'$$

Hence $\boxed{V(\vec{r}) = \dfrac{1}{4\pi R^2}\oint_S V(\vec{r}')\,da' = V_{ave}}$ ← average over sphere

Uniqueness of solutions to electrostatic problems

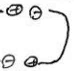


$$\nabla^2 V = -\rho/\varepsilon_0$$

specify ρ

specify V at S: $V|_S = V_S$

(Dirichlet problem)

In the end, discuss Ex 

Consider two solutions

$$\begin{cases} \nabla^2 V_1 = -\rho/\varepsilon_0 \\ V_1|_S = V_S \end{cases}, \quad \begin{cases} \nabla^2 V_2 = -\rho/\varepsilon_0 \\ V_2|_S = V_S \end{cases} \quad \text{let } \delta V = V_2 - V_1$$

$$\begin{cases} \nabla^2 \delta V = 0 \\ \delta V|_S = 0 \end{cases} \qquad 0 = \int_V d\tau\, \delta V \nabla^2 \delta V = \int_V d\tau \left[\nabla\cdot(\delta V \nabla \delta V) - (\nabla \delta V)^2\right]$$

$$= \underbrace{\sum_i \oint_{S_i} d\vec{a}\cdot(\delta V \nabla \delta V)}_{0} - \int_V d\tau (\nabla \delta V)^2$$

$$\Rightarrow \int_V d\tau\, (\nabla \delta V)^2 = 0 \Rightarrow \delta V = C \xrightarrow{\delta V|_S = 0} \delta V \equiv 0$$

---

if instead specify $\frac{\partial V}{\partial n}$: $\left.\frac{\partial V}{\partial n}\right|_S = \left(\frac{\partial V}{\partial n}\right)_S$ (Neumann problem)

Then $\begin{cases} \nabla^2 \delta V = 0 \\ \left.\frac{\partial \delta V}{\partial n}\right|_S = 0 \end{cases}$ still leads to $\nabla \delta V \equiv 0$ and $\underline{\delta V = \text{const.}}$

---

if instead specify total surface charges for <u>conductors</u> $\oint_{S_i} d\vec{a}\cdot \nabla V = -\frac{q_i}{\varepsilon_0}$

Then $\begin{cases} \nabla^2 \delta V = 0 \\ \oint_{S_i} d\vec{a}\cdot\nabla\delta V = 0 \\ \delta V|_{S_i} = C_i \end{cases}$ Then $0 = -\sum_i \frac{C_i\, 0}{\varepsilon_0} - \int_V d\tau (\nabla \delta V)^2$

again $\delta V \equiv \text{const.}$

Green's function

$$\boxed{\nabla^2 G(\vec{r};\vec{r}\,') = \delta_D^{(3)}(\vec{r}-\vec{r}\,')}$$

some boundary condition at S

solution exists!

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Consider Green's identity

$$\int_V d\tau \left(T\nabla^2 U - U\nabla^2 T\right) = \oint_S d\vec{a}\cdot\left(T\nabla U - U\nabla T\right)$$

let $T = V(\vec{r})$, $U = G(\vec{r};\vec{r}\,')$

$$\int_V d\tau \left(V(\vec{r})\underbrace{\nabla^2 G(\vec{r};\vec{r}\,')}_{\delta_D^{(3)}(\vec{r}-\vec{r}\,')} - G(\vec{r};\vec{r}\,')\underbrace{\nabla^2 V}_{-\frac{\rho(\vec{r})}{\varepsilon_0}}\right)$$

$$= \oint_S d\vec{a}\cdot\left(V(\vec{r})\nabla G(\vec{r};\vec{r}\,') - G(\vec{r};\vec{r}\,')\nabla V(\vec{r})\right)$$

$$\Rightarrow \quad V(\vec{r}\,') + \frac{1}{\varepsilon_0}\int_V d\tau\, G(\vec{r};\vec{r}\,')\rho(\vec{r})$$

$$= \sum_i \oint_{S_i} da \left(V(\vec{r})\frac{\partial G(\vec{r};\vec{r}\,')}{\partial n} - G(\vec{r};\vec{r}\,')\frac{\partial V}{\partial n}(\vec{r})\right)$$

① Dirichlet $V\big|_{S_i}$ given, we require $G(\vec{r};\vec{r}\,')\big|_{\vec{r}\in S_i} = 0$

Then $\cancel{V(\vec{r}\,') = -\frac{1}{\varepsilon_0}\int_V d\tau\, G(\vec{r};\vec{r}\,')\rho(\vec{r})} + \oint_{S_i} da\, V(\vec{r})\frac{\partial G(\vec{r};\vec{r}\,')}{\partial n}$

For infinite space $G(\vec{r};\vec{r}\,') = -\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}$

we find $V(\vec{r}\,') = \frac{1}{4\pi\varepsilon_0}\int_V d\tau \frac{\rho(\vec{r})}{|\vec{r}\,'-\vec{r}|}$ familiar result!

② Neumann $\frac{\partial V}{\partial n}\Big|_{S_i}$ given, but we can't require $\frac{\partial G}{\partial n}\Big|_{S_i} = 0$

because $\oint_{S_i} da \frac{\partial G}{\partial n} = \delta_i$ we may require $\frac{\partial G}{\partial n}\Big|_{S_i} = \frac{\delta_i}{S_i}$

(Gauss's Law!)

Then we get $\cancel{V(\vec{r}) = -\frac{1}{\varepsilon_0}\int_V d\tau\, G(\vec{r};\vec{r}')\rho(\vec{r}')} + \langle V\rangle_{S_i}\delta_i$ ← average potential on surface

$$- \oint_{S_i} da\, G(\vec{r};\vec{r}')\frac{\partial V}{\partial n}(\vec{r})$$

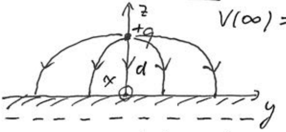
③ Conductor boundary conditions?

$$\begin{cases} V\big|_{S_i} = V_i \text{ (const.)} \\ \oint_{S_i} da \frac{\partial V}{\partial n} = -\frac{q_i}{\varepsilon_0} \end{cases} \quad \text{we require} \quad \begin{cases} G(\vec{r};\vec{r}')\big|_{S_i} = G_i \text{ (const)} \\ \oint_{S_i} da \frac{\partial G}{\partial n} = 0 \end{cases}$$

Then we get $+ G\big|_{S_i}\frac{q_i}{\varepsilon_0}$

So
$$V(\vec{r}) = -\frac{1}{\varepsilon_0}\int_V d\tau\, G(\vec{r};\vec{r}')\rho(\vec{r}')$$
$$+ \sum_{i\in \text{Dirichlet}} \oint_{S_i} da\, V(\vec{r})\frac{\partial G(\vec{r};\vec{r}')}{\partial n}$$
$$+ \sum_{i\in \text{Neumann}} \left(-\oint_{S_i} da\, G(\vec{r};\vec{r}')\frac{\partial V(\vec{r}')}{\partial n} + \underbrace{\langle V\rangle_{S_i}\delta_i}_{\text{Special Comment}}\right)$$
$$+ \sum_{i\in \text{Conductors}} G\big|_{S_i}\frac{q_i}{\varepsilon_0}$$

Method of images



$V(\infty) = 0$

grounded conductor

$V = 0$

consider another problem

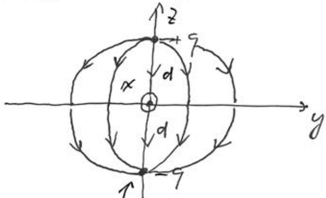


image charge

potential on the xy plane

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

Equipotential

$$V(x, y, 0) = 0$$

Potential in $z>0$ region

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} = -\frac{q}{\varepsilon_0}\delta_D(x)\,\delta_D(y)\,\delta_D(z-d)$$

boundary condition

$$V = 0 \quad \text{at } z = 0 \qquad (\text{Dirichlet})$$

According to uniqueness theorem, the solution of the 2nd problem at $z>0$ is the solution of the first problem.

$$V(x, y, z) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

$$\left.\frac{\partial V}{\partial n}\right|_{z=0} = -\frac{q}{4\pi\varepsilon_0}\left[\frac{(-d)}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$= \frac{qd}{2\pi\varepsilon_0}\frac{1}{(x^2+y^2+d^2)^{3/2}}$$

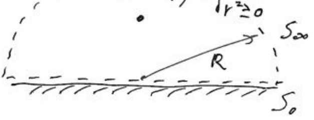
$$\sigma = -\varepsilon_0\frac{\partial V}{\partial n} = -\frac{qd}{2\pi}\frac{1}{(x^2+y^2+d^2)^{3/2}} \qquad \text{Surface charge density}$$

Total charge induced

$$q_{ind} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \ \sigma(x,y) = (2\pi)\int_0^{\infty} r\,dr \left(-\frac{qd}{2\pi}\right)\frac{1}{(r^2+d^2)^{3/2}}$$

$$= -\frac{qd}{2}\int_0^{\infty} \frac{d(r^2)}{(r^2+d^2)^{3/2}} = +qd \left.\frac{1}{(r^2+d^2)^{1/2}}\right|_{r^2=0}^{r^2=\infty}$$

$$\boxed{q_{ind} = -q}$$



Follows from Gauss's Law

$$\underbrace{\oint_{S_\infty} \vec{E}\cdot d\vec{a}}_{=0} - \underbrace{\oint_{S_0} \vec{E}\cdot\hat{z}\,da}_{q_{ind}/\varepsilon_0} = \frac{q}{\varepsilon_0}$$

because as $R\to\infty$, $|\vec{E}| \propto \frac{1}{r^3}$

* Force acting on the positive charge q ?

$$\vec{F}_e = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z} \quad \text{(equivalent to the image charge)}$$

* Electrostatic ~~potential~~ Energy ?

2nd problem: $U = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$

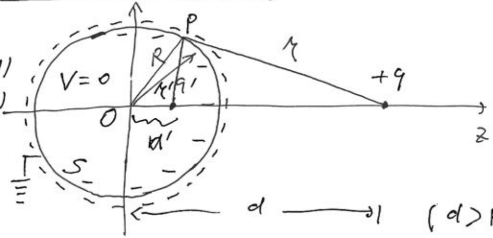
original problem: $U = -\left(\frac{1}{2}\right)\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$

$$W = -\int_{\infty}^{d} \vec{F}_e\cdot d\vec{l} = +\int_{\infty}^{d} \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4z^2}\,dz = -\frac{q^2}{16\pi\varepsilon_0 d}$$

Note: no work is needed to move the induced charge to the xy plane ?

Method of image — Spherical Conductor

$$\begin{cases} \nabla^2 V = -\frac{q}{\varepsilon_0}\delta_D(x)\,\delta_D(y)\,\delta_D(z-d) \\ V\big|_S = 0 \end{cases}$$



$(d > R)$

consider an image charge q' at $z = d'$

if we choose $d' = \frac{R^2}{d} < R$, then $\frac{d'}{R} = \frac{R}{d}$ (inversion point)

triangles $Oq'P$ and OPq are similar triangles

hence $r = \overline{Pq}$, $r' = \overline{Pq'}$, $\frac{r'}{r} = \frac{R}{d}$

if we choose $\frac{q'}{r'} = -\frac{q}{r} \Rightarrow q' = -\frac{r'}{r}q = -\frac{R}{d}q$

Then total potential due to q and q' on the sphere

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right) = 0$$

Solution outside the sphere solves the original problem by uniqueness theorem!

total induced charge $q_{ind} = q' = -\frac{R}{d}q$

Force acting on q: $\vec{F}_e = +\frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')^2}\hat{z} = -\frac{1}{4\pi\varepsilon_0}\frac{Rd\,q^2}{(d^2-R^2)^2}\hat{z}$

* Energy?

image charge: $U = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'} = -\frac{1}{4\pi\varepsilon_0}\frac{R\,q^2}{(d^2-R^2)}$

Original problem?

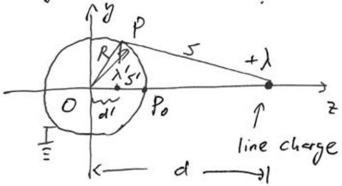
$$W = -\int_\infty^d \vec{F}_e \cdot d\vec{l} = -\int_\infty^d dz \frac{1}{4\pi\varepsilon_0} \frac{Rzq^2}{(z^2-R^2)^2}$$

$$= \frac{-Rq^2}{4\pi\varepsilon_0}\frac{1}{2}\int_\infty^{d^2} \frac{d(z^2)}{(z^2-R^2)^2} = \frac{-Rq^2}{4\pi\varepsilon_0}\frac{1}{2}\left(\frac{1}{z^2-R^2}\right)\Bigg|_{z^2=d^2}^{z^2=\infty}$$

$$= -\underset{\sim}{\frac{1}{2}}\frac{1}{4\pi\varepsilon_0}\frac{Rq^2}{d^2-R^2}$$

Cylindrical problem?

$$\begin{cases} \nabla^2 V = \frac{-\lambda}{\varepsilon_0}\delta_D(y)\delta_D(z-d) \\ V\big|_S = 0 \end{cases}$$



again if $d' = \frac{R^2}{d}$

Then $\frac{\overline{P\lambda}}{\overline{P\lambda'}} = \frac{S}{S'} = \frac{d}{R} = \frac{R}{d'}$

if we pick $V=0$ at P_0 $(y=0,\ z=R)$

Then $V = \frac{1}{2\pi\varepsilon_0}\left(\lambda \ln\frac{S'}{R-d'} + \lambda' \ln\frac{S}{d-R}\right)$

Now $\frac{R-d'}{d-R} = \frac{R-\frac{R^2}{d}}{d-R} = \frac{R}{d} = \frac{S'}{S}$

if we choose $\boxed{\lambda' = -\lambda}$, then

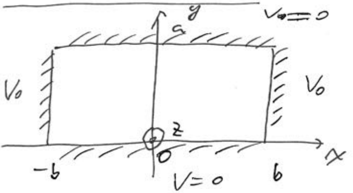
$$V = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{S'(d-R)}{S(R-d')} = 0 \quad \text{on cylinder}$$

By uniqueness theorem, solves the exterior of the cylinder:

$$V(y,z) = \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\frac{\sqrt{y^2+(z-R^2/d)^2}}{R-R^2/d} - \ln\frac{\sqrt{y^2+(z-d)^2}}{d-R}\right]$$

force? electrostatic energy?

Separation of variables



infinitely long "cavity"
bounded by metal plates

$V = V(x, y)$

Laplace's eq.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

boundary condition

$$\begin{cases} V(x, 0) = 0 \\ V(x, a) = 0 \\ V(-b, y) = V_0 \\ V(b, y) = V_0 \end{cases}$$

Consider SoV type solutions

$$V(x, y) = X(x)\, Y(y)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = X'' Y + Y'' X = 0 \Rightarrow \underbrace{-\frac{X''}{X}}_{\text{depends on } x} = \underbrace{\frac{Y''}{Y}}_{\text{depends on } y} = k^2 \ (\text{const.})$$

$$\begin{cases} Y'' + k^2 Y = 0 \\ Y(0) = 0 \\ Y(a) = 0 \end{cases} \Rightarrow \quad Y_n(y) = \sin\frac{n\pi y}{a}, \quad n = 1, 2, 3, \ldots \qquad k_n = \frac{n\pi}{a}$$

$$X'' - \left(\frac{n\pi}{a}\right)^2 X = 0 \Rightarrow \quad X(x) \sim \mathrm{ch}\frac{n\pi x}{a}, \ \mathrm{sh}\frac{n\pi x}{a}$$

Consider linear superposition.

$$V(x, y) = \sum_{n=1}^{\infty} \left(A_n \,\mathrm{ch}\frac{n\pi x}{a} \sin\frac{n\pi y}{a} + B_n \,\mathrm{sh}\frac{n\pi x}{a} \sin\frac{n\pi y}{a} \right)$$

reflection symmetry $V(x, y) = V(-x, y)$, so $B_n = 0$

b.c. at $x = \pm b$

$$V_0 = \sum_{n=1}^{\infty} A_n \,\mathrm{ch}\frac{n\pi b}{a} \sin\frac{n\pi y}{a} \longleftarrow \text{Fourier expansion}$$

How to determine coefficients A_n's ?

Orthonormal relation

$$\int_0^a dy \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a} = \frac{a}{2}\delta_{nm}, \quad n, m = 1, 2, \cdots$$

This means that $\mathrm{ch}\frac{n\pi b}{a} A_n = \frac{2}{a}\int_0^a dy\, V_0 \sin\frac{n\pi y}{a}$

$$= \frac{2V_0}{a}\frac{a}{n\pi}\cos\frac{n\pi y}{a}\Big|_{y=a}^{y=0}$$

$$= \frac{2V_0}{n\pi}\left(1-(-1)^n\right)$$

Hence $$\boxed{V(x, y) = \frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\frac{\mathrm{ch}\frac{n\pi x}{a}}{\mathrm{ch}\frac{n\pi b}{a}}\sin\frac{n\pi y}{a}}$$

* Eigenfunction problem ~ defined by $\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(a) = 0 \end{cases}$
(eigen value)

eigenfunctions $\psi_n(y) = \sqrt{\frac{2}{a}}\sin\frac{n\pi y}{a}, \quad n = 1, 2, 3, \cdots \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2$

$$\psi_n(0) = \psi_n(a) = 0, \quad \psi_n''(y) + \lambda_n\psi_n(y) = 0$$

Consider two eigenfunctions $(\psi_m\psi_n' - \psi_n\psi_m')'$

$$\begin{cases} \psi_n'' + \lambda_n\psi_n = 0 \\ \psi_m'' + \lambda_m\psi_m = 0 \end{cases} \Rightarrow \overbrace{(\psi_m\psi_n'' - \psi_n\psi_m'')} + (\lambda_n - \lambda_m)\psi_n\psi_m = 0$$

integrate over $[0, a]$

$$\begin{cases} \psi_n(0) = \psi_n(a) = 0 \\ \psi_m(0) = \psi_m(a) = 0 \end{cases}$$

$$-(\lambda_n - \lambda_m)\int_0^a dy\, \psi_n\psi_m = \int_0^a dy\,(\psi_m\psi_n' - \psi_m'\psi_n)' = \left[\psi_m\psi_n' - \psi_n\psi_m'\right]_{y=0}^{y=a} = 0$$

Hence $\int_0^a dy\,\psi_n\psi_m \triangleq \langle\psi_n, \psi_m\rangle = 0$ if $\lambda_n \neq \lambda_m$

We can rescale / normalize ψ_n's properly such that

$$\langle \psi_m , \psi_n \rangle = \int_0^a dy\, \psi_m(y)\, \psi_n(y) = \delta_{nm}$$

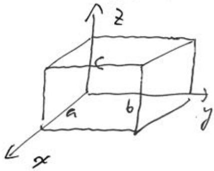
(orthonormal relations)

if we want to find expansion

$$f(y) = \sum_{n=1}^{\infty} C_n \underbrace{\psi_n(y)}$$ ← a complete set of "basis functions"

Then $C_n = \langle \psi_n , f \rangle = \int_0^a dy\, \psi_n(y)\, f(y)$

Comment: What to do with all six faces having non-trivial boundary values?



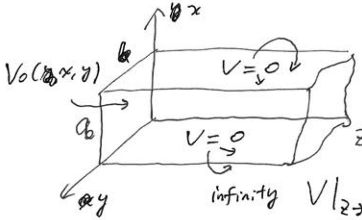
use principle of superposition, since Laplace's eq. is linear.

Ex: More dimensions

semi-infinite region

$z > 0$

$0 < x < a, \quad 0 < y < b$



infinity $V|_{z \to \infty} \to 0$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \quad \text{consider form}$$

$$X(x)\, Y(y)\, Z(z)$$

Then $\displaystyle \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$

$$\begin{cases} X'' + k^2 X = 0 \\ X(0) = X(a) = 0 \end{cases} \qquad \begin{cases} Y'' + l^2 Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

$X_n(x) = \sin \frac{n\pi x}{a}, \quad n = 1, 2, \cdots$

$Y_l(y) = \sin \frac{l\pi y}{b}, \quad l = 1, 2, \cdots$

Then $Z'' - \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2\right] Z = 0, \quad Z \sim e^{\pm\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2}\, z}$

$$V(x, y, z) = \sum_{n=\phi}^{\infty} \sum_{l=\phi}^{\infty} \left(A_{nl}\, e^{-\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2}\, z} + B_{nl}\, e^{+\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2}\, z} \right) \times \sin \frac{n\pi x}{a} \sin \frac{l\pi y}{b}$$

$B_{nl} = 0$ because $V|_{z \to \infty} \to 0$

$$\boxed{A_{nl} = \left(\frac{2}{a}\right)\left(\frac{2}{b}\right) \int_0^a dx \int_0^b dy\, V_0(x, y) \sin \frac{n\pi x}{a} \sin \frac{l\pi y}{b}}$$

Can compute for any specified $V_0(x, y)$ at $z = 0$

SoV — Spherical Coordinates $V = V(r, \theta, \phi)$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial^2 \phi^2}}_{\text{neglected}} = 0$$

Azimuthal symmetry $V = V(r, \theta)$

consider $R(r)\,\Theta(\theta)$ — SoV form

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{\text{only depends on } r} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{\text{only depends on } \theta} = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \lambda\Theta = 0, \quad 0 \le \theta \le \pi$$

if we require $\Theta(\theta)$ is regular at $\theta = 0, \pi$ (poles)

Then $\lambda_\ell = \ell(\ell+1)$, $\ell = 0, 1, 2, \cdots$

$\Theta_\ell(\theta) = P_\ell(\cos\theta)$ ← Legendre polynomials

Rodriguez formula $P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \quad -1 \le x \le 1$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$\vdots$

$$\boxed{\int_0^\pi \sin\theta\, d\theta\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

orthonormal relations

let $x = \cos\theta$

$$\begin{cases} \frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right) + \lambda_\ell P_\ell = 0 \\ \frac{d}{dx}\left((1-x^2)\frac{dP_{\ell'}}{dx}\right) + \lambda_{\ell'} P_{\ell'} = 0 \end{cases}$$

$$-(\lambda_\ell - \lambda_{\ell'}) P_\ell P_{\ell'} = \left[P_{\ell'} \frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right) - P_\ell \frac{d}{dx}\left((1-x^2)\frac{dP_{\ell'}}{dx}\right)\right]$$

$$= \frac{d}{dx}\left[(1-x^2) P_{\ell'} \frac{dP_\ell}{dx} - (1-x^2) P_\ell \frac{dP_{\ell'}}{dx} \right]$$

$$-(\lambda_\ell - \lambda_{\ell'}) \int_{-1}^{1} dx\, P_\ell P_{\ell'} = \left[(1-x^2)\left(P_{\ell'}\frac{dP_\ell}{dx} - P_\ell \frac{dP_{\ell'}}{dx}\right)\right]\Bigg|_{x=-1}^{x=1} = 0$$

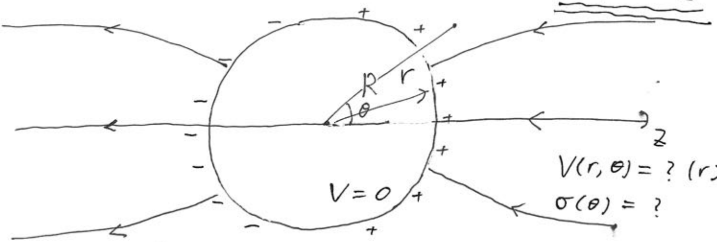
so $\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = 0$

if $\ell \neq \ell'$

Comment: the other solution to $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta = 0$ divergent at $\theta = 0$ and/or $\theta = \pi$

Ex: spherical conductor in a uniform external electric field

$\underline{\vec{E}_{ext} = -E_0 \hat{z}}$



$V(r,\theta) = ?\ (r > R)$

$\sigma(\theta) = ?$

radial equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \ell(\ell+1) R(r) = 0 \Rightarrow R \sim r^\ell, \frac{1}{r^{\ell+1}}$$

so $$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

b.c. at $r = R$

$$0 = \sum_{\ell=0}^{\infty}\left(A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} = \frac{2\ell+1}{2}\int_0^{\pi} \sin\theta \, d\theta \, P_\ell(\cos\theta)\cdot 0 = 0$$

b.c. at $r \to \infty$, $V(r,\theta) \to E_0 z = E_0 r\cos\theta$

$$\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \xrightarrow[r\to\infty]{} E_0 r\cos\theta$$

$$= E_0 r P_1(\cos\theta)$$

$\Rightarrow$ for $\ell \neq 1$, $A_\ell = 0$

for $\ell = 1$, $A_1 = E_0$

Since $A_1 R + \frac{B_1}{R^2} = 0 \Rightarrow B_1 = -E_0 R^3$

Therefore $\boxed{V(r,\theta) = E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$

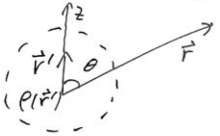
$$\sigma(\theta) = -\varepsilon_0\left(\frac{\partial V}{\partial n}\right)\Bigg|_{r=R} = -\varepsilon_0\left(\frac{\partial V}{\partial r}\right)_{r=R} = -\varepsilon_0 E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\Bigg|_{r=R}$$

Therefore $\boxed{\sigma(\theta) = -3\varepsilon_0 E_0 \cos\theta}$

Electric field can be found by $\vec{E} = -\nabla V$ at $r > R$

Multiple Expansion of potential

localised charge distribution



Spherical coordinates

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\left(1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta\right)^{-1/2}$$

use $\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -(4\pi)\,\delta_D(\vec{r}-\vec{r}') = -(4\pi)\frac{1}{2\pi r'^2}\delta_D(r-r')\,\delta_D(\cos\theta - 1)$

let $\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} P_l(\cos\theta)\left.\begin{cases} \frac{B_l}{r^{l+1}}, & r>r' \\ A_l r^l, & r<r' \end{cases}\right\} R_l(r)$

$\delta_D(\cos\theta - 1) = \sum_{l=0}^{\infty} C_l P_l(\cos\theta) \Rightarrow C_l = \frac{2l+1}{2}\int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\,\delta_D(\cos\theta - 1) = \frac{2l+1}{2}P_l(1) = \frac{2l+1}{2}$

$$\Rightarrow \boxed{\delta_D(\cos\theta-1) = \sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\cos\theta)}$$

$$\sum_{l=0}^{\infty}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR_l}{dr} - \frac{l(l+1)}{r^2}R_l\right)P_l(\cos\theta) = -\sum_{l=0}^{\infty}\frac{2l+1}{r'^2}\delta_D(r-r')\,P_l(\cos\theta)$$

we need $\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR_l}{dr} - \frac{l(l+1)}{r^2}R_l = -\frac{2l+1}{r'^2}\delta_D(r-r')$

integrate from $r=r'-\varepsilon$ to $r=r'+\varepsilon$ $\int_{r'-\varepsilon}^{r'+\varepsilon} r^2 dr$

$$\begin{cases} \left(r^2\frac{dR_l}{dr}\right)\Big|_{r'-\varepsilon}^{r'+\varepsilon} = -(2l+1) \\ R_l\big|_{r'+\varepsilon} = R_l\big|_{r'-\varepsilon} \end{cases} \Rightarrow \begin{cases} -(l+1)\frac{B_l}{r'^{l+2}} - lA_l r'^{l-1} = -\frac{(2l+1)}{r'^2} \\ \frac{B_l}{r'^{l+1}} = A_l r'^l \end{cases}$$

$$\Rightarrow \begin{cases} A_l = \frac{1}{r'^{l+1}} \\ B_l = r'^l \end{cases} \Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}P_l(\cos\theta)\begin{cases}\frac{r'^l}{r^{l+1}}, & r>r' \\ \frac{r^l}{r'^{l+1}}, & r<r'\end{cases}$$

$$= \frac{1}{r_>}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{r_<}{r_>}\right)^l$$

(for $r \gg r'$)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\varepsilon_0}\int d\tau' \rho(\vec{r}') \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d\tau' \rho(\vec{r}') P_l(\cos\theta)\, r'^l$$

$$= \frac{1}{4\pi\varepsilon_0}\Big[\frac{1}{r} \underbrace{\int d\tau' \rho(\vec{r}')}_{q \text{ monopole}} + \frac{\hat{r}}{r^2}\cdot \underbrace{\int d\tau' \rho(\vec{r}')\, \vec{r}'}_{\vec{p} \text{ dipole}}$$

$$+ \frac{1}{r^3}\hat{r}_i \hat{r}_j \underbrace{\int d\tau' \rho(\vec{r}') \frac{3r'_i r'_j - r'^2\delta_{ij}}{2}}_{Q_{ij} \text{ quadrupble}} + \cdots \Big]$$

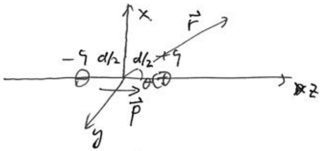
monopole $\quad q = \int d\tau' \rho(\vec{r}') \qquad V_{monopole} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

dipole $\quad \vec{p} = \int d\tau' \rho(\vec{r}')\, \vec{r}' \qquad V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$

quadruple $\quad Q_{ij} = \frac{3}{2}\int d\tau' \rho(\vec{r}')\left(r'_i r'_j - \frac{1}{3}r'^2\delta_{ij}\right)$

$$V_{quad} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

Example: physical dipole



$\vec{p} = (-q)\left(-\frac{d}{2}\hat{z}\right) + q\left(\frac{d}{2}\hat{z}\right) = qd\,\hat{z}$

$|\vec{p}| = qd$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}-\frac{d}{2}\hat{z}|} - \frac{1}{|\vec{r}+\frac{d}{2}\hat{z}|}\right)$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}-\frac{d}{2}\hat{z}}{|\vec{r}-\frac{d}{2}\hat{z}|^3} - \frac{\vec{r}+\frac{d}{2}\hat{z}}{|\vec{r}+\frac{d}{2}\hat{z}|^3}\right)$$

when $|\vec{r}| \gg d \quad |\vec{r} \pm \frac{d}{2}\hat{z}|^{-1} = \left(r^2 \pm d\,r\cos\theta + \frac{d^2}{4}\right)^{-1/2} \simeq$

$$= \frac{1}{r}\left(1 \pm \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2} \simeq \frac{1}{r}\left(1 \mp \frac{d}{2r}\cos\theta\right)$$

$$V(\vec{r}) \xrightarrow{|\vec{r}| \gg d} \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \longleftarrow \text{dipole term } (\ell = 1)$$

dipolar field

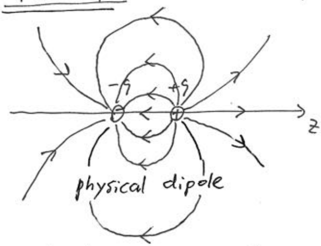
$$E_i = -\partial_i V = -\frac{1}{4\pi\varepsilon_0}\partial_i\left(\frac{p_j r_j}{r^3}\right) = -\frac{p_j}{4\pi\varepsilon_0}\partial_i\left(\frac{r_j}{r^3}\right)$$

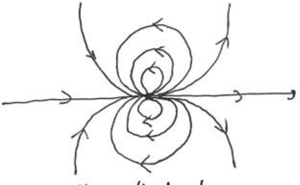
$$= -\frac{p_j}{4\pi\varepsilon_0}\left(\frac{\delta_{ij}}{r^3} - \frac{3r_j}{r^4}\partial_i r\right) = -\frac{p_j}{4\pi\varepsilon_0 r^3}\left(\delta_{ij} - \frac{3r_i r_j}{r^2}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(p_i - \frac{3(p_i r_i) r_j}{r^2}\right)$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)}$$

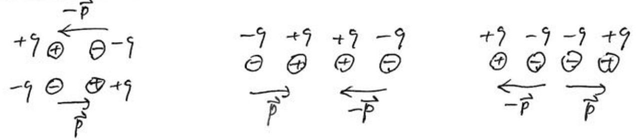
"Pure" dipole $\quad d \to 0,\ q \to \infty.$ but $p = qd$ fixed



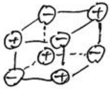


"pure" dipole

quadruple Examples $(\ell = 2)$



Octupole $(\ell = 3)$



Theorem: The lowest order non-zero multipole moment is invariant under a change of coordinate origin!