$$F(q,P,t) = q\cdot P + F_2(q,P,t)$$
$$= q\cdot P + \varepsilon G(q,P,t)$$
$$= q\cdot P + \varepsilon G(q,P,t) + O(\varepsilon^2)$$

$$(q,p) \to (Q,P)$$

Examples:

1) $G = H$ is the generator of infinitesimal time translations

hence if $H$ is time-independent then $H$ is conserved

2) $G = p_i$ is the generator of infinitesimal translations of $q_i$

hence $p_i$ is constant of motion if $H$ is invariant under translations of $q_i$

3) $G = xp_y - yp_x$

$= (\underline{r}\times\underline{p})\cdot\hat{\underline{z}} = L_z$

is the generator of infinitesimal rotations about z-axis

$L_z$ is conserved if $H$ is invariant

under rotations around the z-axis

If $L_x, L_y$ are both constants of motions then $\{L_x, L_y\} = L_z$

so $L_z$ is also constant of motion and so $L$ is constant

$\{L_i, L_j\} = \varepsilon_{ijk} L_k$

where $\varepsilon_{ijk}$ are the structure constant of the Lie algebra $SO(3)$ associated

Recall that under the parametrized family of ICTS with generating function $G(z, \alpha)$

$$\delta z = \delta\alpha \, \underline{\underline{J}} \, \frac{\partial G}{\partial \underline{\underline{z}}}$$

and so far any phase function $W(z, \alpha)$

$$\delta W = d\alpha \left(\frac{\partial W}{\partial \underline{z}}\right)^T \cdot \delta \underline{z}$$

$$= d\alpha \left(\frac{\partial W}{\partial \underline{z}}\right)^T \underline{\underline{J}} \frac{\partial G}{\partial \underline{z}}$$

$$= d\alpha \{W, G\}$$

$$\frac{dW}{d\alpha} = \{W, G\} = -L_G W$$

where $L_G(\cdot) = \{G, (\cdot)\}$ Lie derivative

The properties of $L_G$ follow the properties of Poisson bracket

Linearity $\quad L_{\lambda A + \mu B} = \lambda L_A + \mu L_B$

$\quad L_A(\lambda B + \mu C) = \lambda L_A B + \mu L_A C$ $\Big\}$ $\lambda, \mu$ are constant

Anti-symmetry

Jacobi identity $\quad [L_A, L_B] = L_A L_B - L_B L_A = L_{\{A,B\}}$

It follows that

$$W(z,\alpha) = \underbrace{e^{-\int_0^\alpha d\alpha L_{G(\alpha)}}}_{T} W(z,0)$$

$T$ is the <u>evolution operator</u> that

evolves $W(z,0) = W(z,\alpha)$

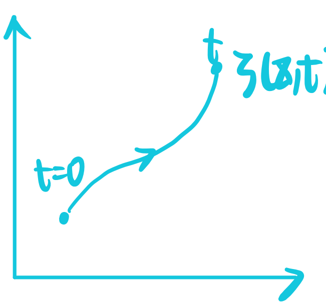
Properties: If $G=H$

$T$ generates time translations

$$T^{-1}(t) = T(-t)$$

$$T^{-1}(t)\,T(t) = Id$$

$$T(t_1)T(t_2) = T(t_1+t_2)$$ Semigroup




$\zeta(z,t) = Tz$ similarly $T\,W(z) = W(\zeta(z))$

For example, $K(\zeta(z)) = H(z)$

becomes $TK = H$

$$K = T^{-1}H$$

Active Picture

Important remark:

$$e^{-\int_0^t dt' L_{H(t')}} = 1 - \int_0^t dt' L_{H(t')}$$

$$+ \int_0^t dt' \int_0^{t'} dt'' L_{H(t')} L_{H(t'')} \cdots$$

$$\int_0^t f(t')dt' \int_0^{t'} g(t'')dt'' + \int_0^t g(t')dt' \int_0^{t'} f(t'')dt''$$

$$= \frac{1}{2}\left(\int f(t')dt'\right)\left(\int g(t'')dt''\right)$$

This provides a generalization of Taylor series

$$f(a+x) = \left(\exp a\frac{d}{dx}\right) f(x)$$

$$= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \cdots$$

Ex Suppose $H$ is autonomous (time-independent)

$$e^{-tL_H} = 1 - tL_H + \frac{1}{2}t^2 L_H^2$$

$$= \mathrm{id} - t\{H, \cdot\} + \frac{1}{2}t^2\{H, \{H, \cdot\}\} - $$

Suppose $H = \frac{p^2}{2m} - max$

particle of mass under constant acceleration $a$

$$x(t) = [\mathrm{id} - t\{H,\cdot\} + \tfrac{1}{2}t^2\{H,\{H,\cdot\}\} - \cdots]x\,|_{t=0}$$

$$\{H,x\} = \frac{\partial H}{\partial x}\frac{\partial x}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial x}{\partial x}$$

$$= -\frac{\partial H}{\partial p} = -\frac{p}{m}$$

$$\{H,x\} = -\frac{p}{m}$$

$$\{H,p\} = \frac{\partial H}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial x}$$

$$p = -m\{H,x\}$$

$$= \frac{\partial H}{\partial x} = -ma$$

$$\{H,\{H,x\}\} = -\frac{1}{m}(-ma) = a$$

$$x(t) = [\mathrm{id} - t\{H,\cdot\} + \tfrac{1}{2}t^2\{H,\{H,\cdot\}\} - \cdots]x\,|_{t=0}$$

$$= x_0 + \frac{p_0}{m}t + \frac{a}{2}t^2$$

where $(x_0, p_0) = (x, t)\,|_{t=0}$

Extended phase space

Suppose n-dof Hamitonian $H(z,t)$ is non-autonomous

Variational Principle

$$\delta\int L\,dt = \delta\int (\underline{p}\cdot\dot{\underline{q}} - H)\,dt$$

time like variable

$$= \delta\int \left(\underline{p}\cdot\frac{dq}{d\xi} - H\frac{dt}{d\xi}\right)d\xi = \delta\int \underline{P}\cdot\frac{d\underline{Q}}{d\xi}\,d\xi$$

this suggest we define $i$

$P_i = p_i \quad Q_i = q_i \quad i = 1,\dots,n$

$Q_{n+1} = \underset{\sim}{t}$

$P_{n+1} = -\underset{\sim}{H}$

$i = n+1$

This change of variation is guaranteed by

$$F_2 = \sum_i^n P_i q_i + P_{n+1} t$$

s.t $\quad p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$

$$K(Q,P) = H(q,p,t) + \frac{\partial F_2}{\partial t} = H(q,p,t) - H$$