Volume 5

Feynman Hughes Lectures

Mathematical Methods in Engineering & Physics

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Notes taken & Transcribed by
John T. Neer

Bookmarks provided

MATH PHYSICS

LECTURE

Hearing Feynman lecture in person on the math-physics is a good preparation for this lecture. The lecture "Relation of Mathematics to physics" can be found at:

http://www.youtube.com/watch?v=M9ZYEb0Vf8U

Anothere related lecture worth hearing is:
Mathematician versus Physicist

http://www.youtube.com/watch?v=obCjODeoLVw

LECTURE I

INTRODUCTION TO THE LECTURES ON MATHEMATICAL TECHNIQUES of ENGINEERING AND PHYSICS

TODAY BEGINS A NEW YEAR of LECTURES AND THIS TIME WE WILL DISCUSS THE SUBJECT of MATHEMATICAL METHODS of ENGINEERING AND PHYSICS. WE WILL BE DISCUSSING SUCH TOPICS AS COMPLEX NOTATION, VECTORS, TENSORS, VARIOUS SPECIAL FUNCTIONS LIKE BESSEL FUNCTIONS WHICH COME ABOUT IN NON LINEAR PROBLEMS, FOURIER TRANSFORMS, AND SERIES.

OUR PROBLEM IS NOT TO FORMULATE THE PHYSICS of A GIVEN PROBLEM BUT RATHER TO APPLY USEFUL MATHEMATICAL TECHNIQUES IN SOLVING THE RESULTING EQUATIONS. AS SUCH OUR EFFORTS WILL NOT BE DIRECTED TOWARD GENERALIZING OUR RESULTS TO A BROAD CLASS of PROBLEMS BUT RATHER TO PAY ATTENTIONS TO CERTAIN ASPECTS of THE PROBLEM IN QUESTION. THE FIRST AND FOREMOST TASK BEFORE US IS TO FORMULATE AN APPROXIMATE EQUATION WHICH CONTAINS THE PHYSICS AND WHICH CAN BE APPROXIMATELY SOLVED. IT IS IMPORTANT TO APPRECIATE THAT WE ARE NOT STRIVING FOR THE EXACT ANSWER BUT ONLY FOR AN APPROXIMATELY RIGHT ANSWER. THIS MUST BE KEPT IN MIND AS WE PROGRESS; ALWAYS REMEMBER THE ACCURACY of THE DESIRED ANSWER.

SOLVING EQUATION

WE ALL KNOW HOW TO SOLVE THE SIMPLE QUADRATIC EQUATION

$$x^2 + 2x + 7 = 0$$

AND WE SHOULD KNOW HOW TO SOLVE THE CUBIC EQUATION

$$x^3 + 3x^2 + x - 1 = 0$$

BUT HOW DO WE SOLVE FOR X GIVEN THE EQUATION

$$e^x = \cos x\,?$$

THESE SIMPLE EXAMPLES CAN BE CATEGORIZED AS A SET OF EQUATIONS WHICH ARE:

(1) LINEAR AND THEREFORE TRIVIAL

(2) QUADRATIC " " "

AND (3) ALL THE REST WHICH ARE NON TRIVIAL

IT IS THE THIRD CATEGORY WHICH INTERESTS THE MOST SINCE IN THE REAL WORLD ALL THE PROBLEM ARE FOUND IN IT.

LET TAKE AN EXAMPLE OF A CUBIC EQUATION AND SEE HOW WE MIGHT SOLVE IT. GIVEN THE CUBIC WHICH WE WRITE AS,

$$\frac{1}{1+x^2} = 2x$$

WE WILL SOLVE IT BY THE TRIAL AND ERROR METHOD. IF THIS METHOD GIVES US THE RIGHT ANSWER - GREAT! THERE IS NOTHING WRONG WITH WRITING DOWN SOME NUMBERS IN THE PROCESS. THIS DOESN'T REPRESENT A CULTURAL LAG; IT IS SICK TO THINK THAT IT IS.

IN ORDER TO HELP US SOLVE THE EQUATION WE WILL INVENT SOME WAYS TO INCREASE OUR EFFICIENCY IN GUESSING THE ANSWER. LETS THEN FORM A TABLE OF VALUES FOR X, THE LEFT HAND SIDE OF THE EQUATION AND THE RIGHT ALSO WE WILL COMPUTE THE DIFFERENCE BETWEEN THE TWO SIDES. THE FOLLOWING STEPS ARE TAKEN

STEP	X	L.H.S	R.H.S	DIFF LHS − RHS
1. TRY X = 0	0	1.000	.000	1.000
2. THAT DIDN'T WORK; TRY X = 1	1	0.50	2.00	−1.500
3. THAT DIDN'T WORK EITHER BUT WE ARE ON EITHER SIDE OF THE ANSWER. WE MIGHT GUESS THE ANSWER LIES .4 OF THE WAY BETWEEN 0 AND 1 SO LET'S TRY X = .5	.5	.8	1.00	−.200
4. WE'RE GETTING BETTER TRY X = .4	.4	.862	+.800	+.062
5. SINCE THE DIFFERENCE IN 3 AND 4 IS ON EITHER SIDE OF THE ANSWER INTERPOLATE BETWEEN THEM, I.E. $\Delta = \frac{62}{262} = .24$ SO TRY X = .424	.424	.847	.848	.001

AND NOW WE HAVE THE ANSWER TO AN ACCURACY OF .1%. SO THIS METHOD IS PRETTY ACCURATE AND IS BETTER THAN A MACHINE BECAUSE IT CAN'T GUESS WHAT TO DO NEXT.

THIS METHOD of INTERPOLATION works well when The Difference between The Two sides of The equation come out to be + and −. When That happens you can interpolate and go again. One word of caution never use graph paper; it is always easier to use the numbers.

There are two more methods which are more appropriate of machine than by hand; they are the method of iteration and Newton's method. The idea is to write the equation as

$$X_{OUT} = \frac{1}{2} \quad \frac{1}{1+X_{IN}^2}$$

where you now try an X_{IN} value, e.g $X_{IN} = 0$ and find X_{OUT}; then plug that value back in, etc.

STEP	X_{IN}	X_{OUT}
1	0	.50
2	.5	.4
3	.4	.431
4	.431	.426

If you want a lot of accuracy in the answer this method has an error which decreases much more slowly than the previous method. Also you have to be careful that the equation is written in the right form otherwise the answer won't converge. To show this solve for X_{IN}

$$X_{IN} = \sqrt{\frac{1}{2X_{OUT}} - 1}$$

Make a table and start evaluating

X_{IN}	X_{OUT}
.4	.5
.5	0.0
0	∞

TO SEE WHY THE ANSWER IS DIVERGING LETS SUBSTITUTE

$$X_{IN} = X_{TRUE} + \epsilon_{IN}$$

Where $$X_T = \sqrt{\frac{1}{2X_T} - 1}$$

Then $$\sqrt{\frac{1}{2(X_T + \epsilon_{IN})} - 1} = \sqrt{\left(\frac{1}{2X_T - 1}\right) - \frac{\epsilon_{IN}}{4X_0^2}} = X_0\sqrt{1 + \frac{\epsilon_{IN}}{4X_0^4}}$$

EXPANDING WE HAVE $$X_0 - \frac{\epsilon_{IN}}{8X_0^3}$$

SO THAT $$\epsilon_{OUT} = -\frac{1}{8X_0^3}\epsilon_{IN}$$

ϵ_{OUT} IS THEN GREATER THAN TWICE THE ERROR IN SO THE ANSWER IS DIVERGING

NEWTON'S METHOD

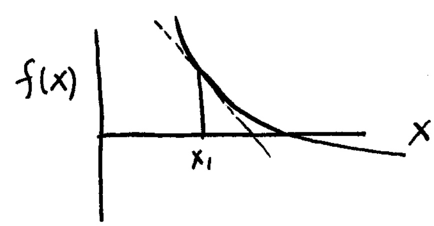
· THE NEWTON'S METHOD REQUIRES WRITING THE FUNCTION IN THE FORM

$$f(X) = 0 \quad , \text{ I. E}$$

$$f(x) = \frac{1}{1+x^2} - 2x = 0$$

IF YOU ARE CLOSE TO THE ANSWER WITH A GUESS, SAY X_1 THEN EVALUATE $f(X_1)$ AND ALSO $f'(X_1)$. IN THIS CASE

$$f'(X) = \frac{2x}{(1+x^2)^2} - 2$$



THEN THE NEXT TRY WOULD BE

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$$

TRY $X_1 = 0$ THEN $f(X_2) = 0 - 1/2 = .5$. AND $f'(X_1) = \frac{.2}{2.64}$

AND $$X_2 = .5 - \frac{.2}{2.64} = .424$$

THIS TECHNIQUE DOESN'T REQUIRE MUCH INTELLIGENCE SO IT IS GREAT FOR A COMPUTER. THE ANSWER WILL ALWAYS CONVERGE EXCEPT FOR A FEW RARE EXAMPLES. HOWEVER THIS METHOD REQUIRES EVALUATING BOTH $f(X)$ AND $f'(X)$. $f'(X)$ MAY BE HARD TO COMPUTE AND DIFFICULT TO EVALUATED.

$\boxed{\text{PROBLEM}}$ SOLVE $e^{-x} = \cos x$ TO 1 %

A word ON COMPLEX ROOTS. If YOU MADE A MISTAKE AND DIDN'T DO THE PROBLEM RIGHT OR YOU DIDN'T EXPECT THE PHYSICS RIGHT THE ROOTS MAY BE COMPLEX. TO FIND COMPLEX ROOTS YOU HAVE TO SOLVE TWO EQUATIONS IN 2 UNKNOWNS. THIS CAN BE DONE BY THE SAME PROCEDURE AS OUTLINED BEFORE. GIVEN $f(x,y) = 0$ AND $g(x,y) = 0$ FIND y AS A FUNCTION OF x IF POSSIBLE THEN SUBSTITUTE BACK INTO ONE OF THE EQUATIONS.

SERIES

WE WANT TO DEAL WITH THE PROBLEM OF SUMMING SERIES. THERE ARE LOTS OF WAYS TO DO IT; THE EASIEST WAY INVOLVES ADDING THE NUMBERS. THIS MAY STRIKE YOU AS ODD SINCE YOU MAY HAVE BEEN TAUGHT A LOT OF SHARP METHOD OF SUMMING SERIES WHICH YOU HAVE ALL FORGOTTEN. YOU CAN OF COURSE TRY TO MEMORIZE THE ANSWER IN SOME SPECIAL CASES AND THAT IS GOOD SOMETIME. FOR EXAMPLE IT IS QUITE USEFUL TO KNOW THAT

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

ONCE I KNOW THAT SERIES I CAN DEAL WITH MORE COMPLICATED SERIES SUCH AS

$$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots = \;?$$

FIRST I CAN SUBSTITUTE FOR $\cos\theta$,

$$\cos\theta = \operatorname{Re} e^{i\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

THEN I HAVE

$$1 + \tfrac{1}{2} a e^{i\theta} + \tfrac{1}{2} a e^{-i\theta} + \tfrac{1}{2} a^2 e^{2i\theta} + \tfrac{1}{2} a^2 e^{-2i\theta} + \cdots =$$

$$\tfrac{1}{2}\left(1 + a e^{i\theta} + a^2 e^{2i\theta} + \cdots\right) + \tfrac{1}{2}\left(1 + a e^{-i\theta} + a^2 e^{-2i\theta} + \cdots\right) =$$

$$\frac{1}{2}\,\frac{1}{(1 - a e^{i\theta})} + \frac{1}{2}\,\frac{1}{(1 - a e^{-i\theta})} = \frac{1}{2}\left[\frac{1 - a\cos\theta + i a\sin\theta}{(1 - a\cos\theta)^2 + (a\sin\theta)^2} + \frac{1 - a\cos\theta - i a\sin\theta}{(1 - a\cos\theta)^2 + (a\sin\theta)^2}\right]$$

$$= \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}$$

A KNOWLEDGE OF COMPLEX ~~ROO~~ NUMBERS IS VERY IMPORTANT IN SIMPLYING GEOMETRIC SERIES. LIKEWISE IT IS IMPORTANT TO BE ABLE TO EXTEND ONE FORMULA TO MORE APPLICATION, E.G THE SERIES

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

IF I NOW HAVE A NEW PROBLEM WHERE I WANT THE SUM OF THE SERIES

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^2}{4} + \cdots = S(x)$$

IF THE NUMBERS IN THE DENOMINATOR ARE NICE NUMBERS I CAN SUM THE SERIES BUT TO DO THAT I NEED TO KNOW HOW TO DIFFERENTIATE AND INTEGRATE SERIES. UNTIL I GET THE THE SERIES IN A FORM I RECOGNIZE. IN THE CASE ABOVE I WANT TO GET RID OF 2, 3, 4, ····. TO DO THAT DIFFERENTIATE WITH RESPECT TO (WRT) X

$$\frac{d}{dx}\left[x\,S(x)\right] = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$1 - x^2 + x^3 - x^4 + \cdots = \frac{1}{1+x}$$

$$\frac{1}{1+x} = \frac{d}{dx}\left[x\,S(x)\right]$$

OR

$$x\,S(x) = \ln(1+x) + \text{CONSTANT}$$

TO FIND THE CONSTANT LET $x=0$ THEN $x\,S(x) = 0$ SO $C = 0$ AND I HAVE

$$S(x) = \frac{1}{x}\ln(1+x)$$

SUPPOSE WE MAKE UP THE SERIES

$$x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots = T(x)$$

AND WE WANT TO FIND $T(x)$. BY DIFFERENTIATING

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{3} + \cdots = T'(x)$$

FROM THE CASE ABOVE

$$T'(x) = -\frac{1}{x}\ln(1-x)$$

NOW WE HAVE TO INTEGRATE,

$$T(x) = \int_0^x -\frac{1}{y}\ln(1-y)\,dy \quad + \cancel{C}^{\,0}$$

SUBSTITUTE $1-Y = e^{-u}$ THEN

$$T(x) = \int \frac{-u\,e^{-u}}{1-e^{-u}}\,du = \int_0^\infty \frac{u\,du}{e^u-1}$$

SO AFTER ALL OUR WORK WE ARE STUCK WITH A INTEGRAL WHICH WE HAVE TO LOOK UP IN A TABLE.

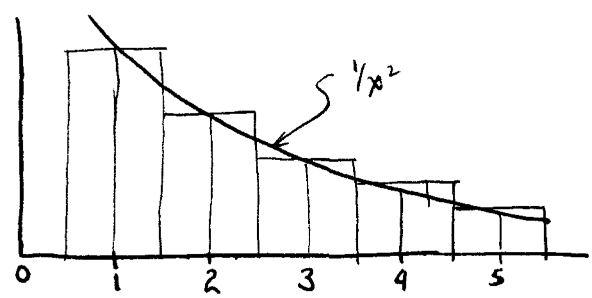
THE ONLY OTHER RECOURSE TO SOLVING THE PREVIOUS PROBLEM IS TO ADD UP THE NUMBERS. THAT SOUNDS DIRTY BUT BELIEVE ME THERE IS NOTHING WRONG WITH IT. WE HAVE THEN FOR $T(1)$

$$1 + 1/4 + 1/9 + 1/16 + \cdots =$$

$$1 + .250 + .111 + .063 + .040 + .028 + .016 + .012 + .010 + \cdots$$

AND WE HAVE THE ANSWER TO 1% ACCURACY. I CAN ADD UP THE SERIES MORE RAPIDLY IF CONSIDER THE SERIES $\sum_{n=1}^{\infty} 1/n^2$.

ONE WAY TO TREAT THIS SERIES IS LIKE THE ADDITION OF A LOT OF RECTANGLES OF 1/2 INTEGER IN WIDTH AND CONSIDER A CONTINUUM OF VALUES. IF I GO TO THE 3RD NUMBER AND INTEGRATE THE REMAINING VALUES I HAVE



$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \int_{3\frac{1}{2}}^{\infty} \frac{dx}{x^2}$$

NOTE THE LOWER INTEGRATION LIMIT IS THE MIDDLE POSITION OF THE LAST INTERVAL TAKEN. LET ME COMPARE THE ACCURACY OF THIS APPROXIMATION WITH THE CRUDEST, I.E ONLY 2 TERMS

$$1 + \frac{1}{2^2} + \int_{2\frac{1}{2}}^{\infty} \frac{dx}{x^2} = 1.250 + \frac{1}{2\frac{1}{2}} = 1.650$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \int_{3\frac{1}{2}}^{\infty} \frac{dx}{x^2} = 1.361 + \frac{1}{3\frac{1}{2}} = 1.647$$

GOOD TO 1% ALREADY

NOTE $\frac{1}{7} = .142857142857$

AND THE NEXT TERM IS

$$1 + 1/2^2 + 1/3^2 + 1/4^2 + \underbrace{\int_{4\frac{1}{2}}^{\infty} \frac{dx}{x^2}}_{2/9} = 1.6458$$

IT IS GOOD TO KNOW THE FOLLOWING SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

$$\ln(1-x)$$

$$\tan^{-1} x$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n+1} \frac{x^{2n+1}}{(2n-1)}$$

$$(1+x)^n$$

SOME PROBLEMS

if $S(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

find $1 - 1/3 + 1/5 - 1/7 + \cdots$

CALL $S = 1 + 1/4 + 1/9 + 1/16 + 1/25 + \cdots$

find $1 + 1/9 + 1/25 + 1/81 = \sum$ odd NUMBER

SUM $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots$

SUM $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots =$

PROBLEM SOLVE $e^{-x} = \cos x$ TO 1%

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$e^{-x} = \cos x \to 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots = \cancel{1} - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\frac{x^4}{24} + \frac{x^3}{6} - \frac{x^2}{2} + x = 0$$

$$x\left(\frac{x^3}{24} + \frac{x^2}{6} - \frac{x}{2} + 1\right) = 0$$

$x = 0$

$$x^3 + 4x^2 - 12x + 24 = 0 \quad = f(x)$$

$$x^2(x+4) - 12x + 24 = 0$$

$$x^2(x+4) = 12x - 24 \qquad x = \sqrt{\frac{12(x-2)}{(x-4)}}$$

$$x^2 = \frac{12(x-2)}{x+4}$$

$$f'(x) = 3x^2 + 4x - 12 = 0$$

TRY $x =$ [illegible] $\quad x^2 = \frac{12(x-2)}{x+4}$

x	LHS	RHS
0	0	-24
2	4	1.5
3/2		12/11

TRY $x_1 = 1$

$$x_2 = 1 - \frac{17}{5} = 1 - 3.4 = -2.4$$

$$x_3 = -2.4 - \frac{60.8}{14.8}$$

LECTURE 3

METHODS of DIFFERENTIATING AND INTEGRATING

THE IS A STRAIGHTforward WAY TO DIfferentiate complicated functions. Suppose we have the function,

$$f(x) = \frac{(1+x^2)^{1/3}}{(1+\cos x)^{3/2}} \frac{\ln x}{x^2}$$

And we want $f'(x)$. Here's what you do:

STEP

1. Write down the function and beside it put a bracket

$$\frac{(1+x^2)^{1/3}}{(1+\cos x)^{3/2}} \frac{\ln x}{x^2} \left\{ \text{Sum of Derivative of Terms} \right\}$$

2. Start differentiating each factor one at a time by first writing down the exponent, then in the denominator write the function that is being differentiated, in the numerator put the derivative of the function. It goes like this

$$\frac{(1+x^2)^{1/3}}{(1+\cos x)^{3/2}} \frac{\ln x}{x^2} \left\{ \underset{\uparrow\ \text{EXPONENT}}{\frac{1}{3}} \frac{\overset{\text{DERIVATIVE}}{2x}}{\underset{\uparrow\ \text{WHAT IS BEING differentiated}}{(1+x^2)}} + 1 \cdot \frac{1/x}{\ln x} - \frac{3}{2} \frac{(-\sin x)}{(1+\cos x)} - 2 \frac{1}{x} \right\}$$

3. Combine the bracket term and simplify if possible

The reason way this technique works can be explained rigourously but quite simply it is associated with the derivative of logarithm. If I have the function

$$f(u, v, w) = u^a v^b w^c \quad \text{And I wand } f'(\ ) \text{ Then}$$

$$\frac{d}{dx}(u^a v^b w^c) = u^a v^b w^c \left(\frac{a}{u} \, du/dx + \frac{b}{v} \frac{dv}{dx} + \frac{c}{w} \frac{dw}{dx} \right)$$

where I use the fact that

$$\frac{d \ln N}{dx} = \frac{1}{N} \frac{dN}{dx}$$

I USE THIS TECHIQUE EVERY TIME I HAVE TO DIFFERENTIATE EVEN WHEN IT IS A SIMPLE FUNCTION LIKE

$$f^{*}(x) = \frac{x^2}{1+x^2} \quad \text{where} \quad f'(x) = \frac{x^2}{1+x}\left(\frac{2}{x} - \frac{1}{1+x}\right)$$

I DO IT THAT WAY BECAUSE I CAN NEVER REMEMBER THE RULE FOR DIFFERENTIATING THE PRODUCT OF TWO FUNCTION, I.E

$$\frac{d\,^{u}/_{v}}{dx} = \frac{u}{v}\left(\frac{du}{u} - \frac{dv}{v}\right) = \frac{1}{v}\,du - u\frac{dv}{v}$$

THIS METHOD OF DIFFERENTIATING IS NOT DISCUSSED IN VERY MANY BOOKS BUT I RECOMMEND YOU LEARN HOW TO USE IT BECAUSE IT IS QUITE VALUABLE.

METHODS OF INTEGRATION

THERE ARE SEVERAL WAYS IN WHICH COMPLICATED INTEGRALS CAN BE EVALUATED; THEY ARE:

(1). BY SUBSTITUTION OF VARIABLES
(2) INTEGRATION BY PARTS
(3). BY COMPLEX VARIABLES

THE THIRD METHOD IS THE ONE I WANT TO WORK WITH BECAUSE IT IS THE MOST POWERFUL METHOD FOR HANDLING COMPLICATED INTEGRALS.

THE SUBSTITUTION WHICH I WILL BE MAKING, WHICH I PRESUME YOU ALL KNOW IS

$$e^{i\theta} = \cos\theta + i\sin\theta$$

ONE OF THE BUILDING BLOCKS WHICH I NEED IS THE FOLLOWING INFORMATION:

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$$

NOW SUPPOSE I WANT TO EVALUATE THE DEFINITE INTEGRAL

$$\int_0^\infty e^{-ax}\cos bx\,dx$$

WHAT DO I DO?

THE FIRST THING IS TO SUBSTITUTE FOR $\cos bx$, $\dfrac{e^{ibx}+e^{-ibx}}{2}$
THEN

$$I = \int_0^\infty e^{-ax}\cos bx\, dx = \int_0^\infty e^{-ax}\left(\frac{e^{ibx}+e^{-ibx}}{2}\right) dx$$

REARRANGING WE GET

$$\frac{1}{2}\int \left[e^{-(a-ib)x} + e^{-(a+ib)x} \right] dx$$

AND RECALLING THAT

$$\int e^{-(\text{SOMETHING})x}\, dx = \frac{1}{\text{SOMETHING}}$$

WE HAVE THAT

$$I = \frac{1}{2}\left[\frac{1}{a-ib} + \frac{1}{a+ib}\right] = \frac{a}{a^2+b^2}$$

ONE NOTE THAT THIS TECHNIQUE ALSO WORKS ON THE INDEFINITE INTEGRAL AS WELL, I.E.

$$\int_0^y e^{-ax}\cos bx\, dx$$

WE WILL OFTEN TIMES HAVE TO DIFFERENTIATE UNDER THE INTEGRAL SIGN BEFORE GETTING THE INTEGRAND INTO A FORM THAT WE CAN EASILY INTEGRATE. THE GENERAL FORMULA THAT WE WANT TO USE IS

$$I = \int_{x_1(\alpha)}^{x_2(\alpha)} f(x,\alpha)\, dx$$

THE SIMPLEST CASE IS WHEN x_1 AND x_2 ARE FIXED VALUES AND DO NOT DEPEND ON THE VARIABLE α. IN THIS CASE

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\left[\int_{x_1}^{x_2} f(x,\alpha)\, dx\right] = \int_{x_1}^{x_2} \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx$$

IN THE OTHER CASE WHEN THE LIMITS OF INTEGRATION DEPEND ON α WE HAVE THAT

$$\frac{d}{d\alpha} I = \int_{x_1(\alpha)}^{x_2(\alpha)} \left[\frac{\partial f(x,\alpha)}{\partial\alpha} + f(x_2(\alpha)) \frac{\partial x_2(\alpha)}{\partial \alpha} - f(x_1(\alpha)) \frac{\partial x_1(\alpha)}{\partial\alpha} \right] dx$$

AS AN EXAMPLE

$$I = \int_0^\infty x e^{-ax} \, dx = -\frac{d}{da} \int_0^\infty e^{-ax} dx = -\frac{d}{da}\left(\frac{1}{a}\right) = +\frac{1}{a^2}$$

LIKEWISE

$$I = \int x^2 e^{-ax} dx = +\frac{d^2}{da^2} \int e^{-ax} dx = \frac{d^2}{da^2}\left(\frac{1}{a}\right) = \frac{2}{a^3}$$

SINCE WE ARE WORKING WITH METHODS OF MATHEMATICS WE WANT TO LEARN TO EXPAND OUR NEW KNOWLEDGE TO NEW PROBLEMS. SUPPOSE I GAVE YOU THE FOLLOWING INTEGRAL TO EVALUATE

$$\int_0^\infty \frac{\sin x}{x} \, dx$$

WHAT WOULD YOU DO? WELL, TRY PUTTING α IN THE SINE FUNCTION SO THAT

$$I(\alpha) = \int_0^\infty \frac{\sin \alpha x}{x} \, dx$$

NOW

$$I'(\alpha) = \int_0^\infty \left[x \frac{\cos \alpha x}{x} \right] dx = \int_0^\infty \cos \alpha x \, dx$$

UNFORTUNATELY WE CAN'T EVALUATE THE LAST INTEGRAL SINCE THE FUNCTION $\cos \alpha x$ IS OSCILLATORY WITH A ZERO AVERAGE VALUE. LET'S TRY A DIFFERENT FUNCTION

$$I = \int_0^\infty e^{-ax} \frac{\sin bx}{x} \, dx$$

$$\frac{dI}{db} = \int_0^\infty e^{-ax} \frac{\cancel{x} \cos bx}{\cancel{x}} \, dx = \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

THEN

$$I = \int \frac{a}{a^2+b^2} \, db = \tan^{-1} \frac{b}{a} + C$$

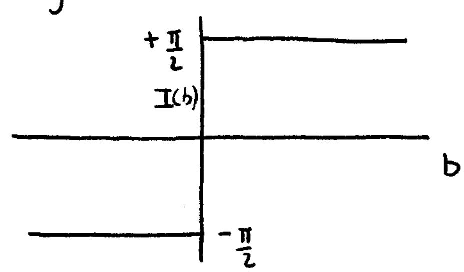
SO FINALLY WE HAVE THAT

$$\int_0^\infty e^{-ax} \frac{\sin bx}{x}\, dx = \tan^{-1}\frac{b}{a} + C$$

TO FIND C CHOOSE $b=0$ WHICH IMPLIES $C=0$. WE ARE NOW IN A POSITION TO FIND $\int_0^\infty \frac{\sin bx}{x}\, dx$ IF WE LET $a=0$ THEN

$$\int_0^\infty \frac{\sin bx}{x}\, dx = \tan^{-1}\infty = \frac{\pi}{2}$$

THUS IF $b<0$ $\int \frac{\sin bx}{x}\, dx = -\frac{\pi}{2}$, IF $b>0$ $\int = +\frac{\pi}{2}$ AND IF $b=0$ $\int = 0$. GRAPHICALLY THE FUNCTION LOOKS LIKE



DELTA FUNCTIONS

WE TRIED TO EVALUATE THE INTEGRAL $\int_0^\infty \cos\beta x\, dx$ AND CONCLUDED THAT IT WAS INDETERMINANT. THIS IS NOT STRICTLY TRUE BECAUSE WE CAN EVALUATE THE INTEGRAL TO BE

$$\int_0^\infty \cos\beta x\, dx = \pi\,\delta(\beta)$$

WHERE $\delta(\beta)$ IS CALLED A DELTA FUNCTION. THE CONCEPT OF A DELTA FUNCTION IS QUITE USEFUL IN PHYSICS AND THEREFORE WORTH DISCUSSING HERE.

THE DELTA FUNCTION CONCEPT INVOLVES A SEQUENCE OF FUNCTIONS CENTERED ABOUT A POINT AND WHOSE AREA EQUALS UNITY. THAT IS THE WIDTH IS PROPORTIONAL TO X AND THE HEIGHT INVERSELY PROPORTIONAL TO X, IN THE LIMIT THE DELTA FUNCTION IS ZERO EVERYWHERE EXCEPT AT $X=0$; THERE IT HAS AN AREA EQUAL TO UNITY,

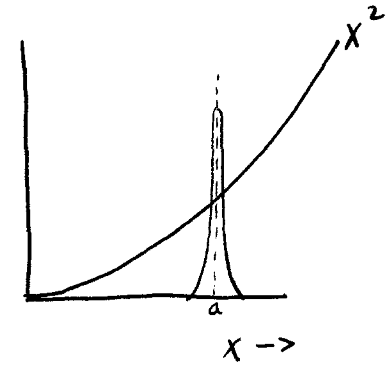
$$\int_{-\infty}^\infty \delta(x)\, dx = 1$$

THE GENERAL FORMULA FOR THE DELTA FUNCTION IS

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

AS AN EXAMPLE LET $f(x) = x^2$, THEN

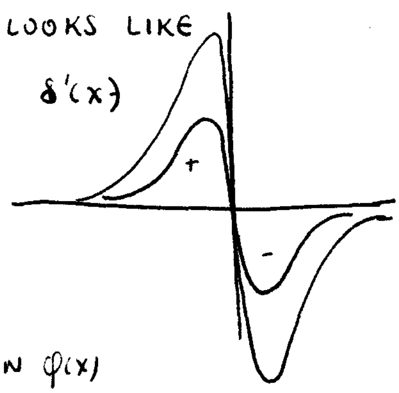
$$\int_{-\infty}^{\infty} x^2 \delta(x-a)\,dx \approx a^2 \int_{-\infty}^{\infty} \delta(x-a)\,dx \simeq a^2$$



SOME IMPORTANT PROPERTIES OF THE DELTA FUNCTION ARE THE FOLLOWING,

$$\int f(x)\,\delta'(x-a)\,dx = -f'(a)$$

THE DERIVATIVE OF THE DELTA FUNCTION LOOKS LIKE



$$\delta(ax) = \frac{1}{|a|}\,\delta(x)$$

AND

$$\delta(x) = +\,\delta(-x)$$

WE CAN EASILY PROVE $\delta(ax) = \frac{1}{|a|}\delta(x)$ BY MULTIPLYING BY AN ARBITRARY FUNCTION $\varphi(x)$ AND THEN INTEGRATING

$$\int \varphi(x)\,\delta(ax)\,dx = \frac{1}{|a|}\int \varphi(x)\,\delta(x)\,dx = \frac{1}{|a|}\,\varphi(0)$$

OTHER DELTA FUNCTION PROPERTIES ARE

$$\delta'(-x) = -\delta'(x)$$
$$x\,\delta(x) = 0$$
$$x\,\delta'(x) = -\delta(x)$$
$$\delta(g(x)) = \frac{1}{g'(x_0)}\,\delta(x-x_0)$$

IN THE PROBLEM $\int_0^{\infty} \cos\beta x\,dx = \pi\,\delta(\beta)$ AND STRAIGHTFORWARD WAY TO EVALUATE THE INTEGRAL IS TO COMPUTE THE INDEFINITE VALUE $\int_0^L \cos\beta x\,dx = \frac{1}{\beta}\sin\beta x\Big|_0^L = \frac{\sin\beta L}{\beta}$ AND LET $L \to \infty$ SET $\sin\beta L = 0$ SINCE FOR ALL PHYSICAL SYSTEMS $\sin\beta L$ WILL EVENTUALLY DAMP OUT.

As another example, I was once given the following integral to evaluate

$$\int_0^{\pi/2} \cos(m \tan\theta)\, d\theta$$

To start this problem set $x = \tan\theta$ then $d\theta = \frac{x}{1+x^2}$

$$\int \to \int_0^\infty \frac{\cos mx}{1+x^2}\, dx = S(m)$$

Differentiating

$$S'(m) = -\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$$

Again

$$S''(m) = -\int_0^\infty x^2 \frac{\cos mx}{1+x^2}\, dx$$

Now I have built a polynomial into the numerator by differentiating. By adding 1 inside the $\int$ sign I have

$$\int_0^\infty \frac{(1+x^2)}{(1+x^2)} \cos mx\, dx = S(m) - S''(m)$$

which becomes the following differential equation

$$S''(m) - S(m) = -\pi\,\delta(m)$$

To solve the differential equation you have to break the region of m into two parts; $m < 0$ and $m > 0$

for $m > 0$ $\quad S''(m) - S(m) = 0$

" $m < 0$ $\quad S''(m) - S(m) = 0$

The solution to the equations are

for $m > 0$ $\quad S(m) = Ae^{m} + Be^{-m}$

" $m < 0$ $\quad S(m) = Ce^{m} + De^{-m}$

We don't know what A, B, C, and D are so to go further we have to use the $\delta(m)$ function to relate the two regions. This shows you how complicated these integrations can get.

FEYNMAN ON THE THEORY of QUARKS - 10/12/70

TODAY I MAY have discovered A fundAMENTAL relationship between SPIN STATISTICS AND QUARK THEORY. There has been A CONTINUING Effort TO TRY TO relate both fermi And Bose SPIN STATISTICS IN QUANTUM MECHANICS. Fermi STATISTICS SAYS THAT SPIN 1/2 PARTICLES DON'T LIKE TO BE IN THE SAME STATE. While BOSE STATISTICS (BOSONS) LIKE TO BE IN THE SAME STATE; These Are THE INTEGRAL SPIN PARTICLES LIKE PHOTONS.

NOW QUARKS AND Theoretically SPIN 1/2 PARTICLE. THREE QUARKS AMAKE UP The fundAMENTAL PARTICLES, E.g. ELECTRONS PROTONS. SUPPOSEDLY AS SPIN 1/2 PARTICLES The OBEY fermi STATISTICS. PROTONS however SEEM TO have quarks which obey Bose STATISTICS.

I NOW BELIEVE THAT ALL The APPARENT ANOMALIES IN QUARK Theory CAN be EXPLAINED If WE CONSIDER QUARKS TO BE PIECES of MATTER WITH SPIN 1/2 THAT OBEY BOSE STATISTICS.

LECTURE 3

PROBLEMS

(1) EVALUATE $\displaystyle\int_0^\infty \frac{\sin x^2}{x^2}\,dx$

(2) SHOW $\displaystyle\int_0^\infty \frac{e^{-ay}-e^{-by}}{y}\,dy = \ln(a/b)$

(3) GIVEN $\displaystyle\int_0^\infty e^{-y^2}\,dy = \sqrt{\pi}/2$

FIND $\displaystyle\int_0^\infty y^2 e^{-y^2}\,dy = ?$

HINT: PUT a INTO THE FIRST $\int$ THEN DIFFERENTIATE

(4). FIND $\displaystyle\int_0^\infty e^{-\beta x^2 - \alpha/x^2}\,dx$

(5). PROVE $\displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}\,e^{-\frac{b^2}{4a^2}}$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

$$I = \int_0^\infty \frac{\sin x^2}{x^2}\,dx = \int \frac{\sin \alpha x^2}{x^2}\,dx$$

$$\frac{dI(\alpha)}{d\alpha} = \int \frac{2x^2 \cos \alpha x^2}{x^2}\,dx \qquad \int \cos \alpha x^2 dx = \frac{1}{2}\sqrt{\pi/2}$$

$$\frac{d^2 I(\alpha)}{d\alpha^2} = -\int 4x^2 \frac{\sin \alpha x^2}{x^2}\,dx = -4 I(\alpha)$$

$$\frac{d^2 I(\alpha)}{d\alpha^2} + 4 I(\alpha) = 0$$

$$I(\alpha) = e^{+2i\alpha} \qquad \dot{I}(\alpha) = +2ie^{+2i\alpha} \qquad \ddot{I} = -4e^{2i\alpha}$$

$$I = e^{2i} = \cos 2$$

$$I = \int_0^\infty \frac{e^{-ay} - e^{-by}}{y}\,dy = \int_0^\infty \frac{e^{-ay}}{y}dy - \int_0^\infty \frac{e^{-by}}{y}\,dy$$

$$I_1(a) = \int_0^\infty \frac{e^{-ay}}{y}\,dy \qquad I'(a) = -\int \frac{y\,e^{-ay}}{y}\,dy = -\frac{1}{a}$$

$$I_2(b) = \int_0^\infty \frac{e^{-by}}{y}\,dy \qquad I'(b) = -\int \frac{y\,e^{-by}}{y}\,dy = -\frac{1}{b}$$

$$I_1(a) = -\ln(a)$$

$$I_2(b) = -\ln(b)$$

$$I = +\ln(a) + \ln(b) = \ln(a/b)$$

GIVEN $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$ find $\int_0^\infty y^2 e^{-y^2} dy$

$I(a) = \int_0^\infty e^{-ay^2} dy$

$I'(a) = \int -y^2 e^{-ay^2} dy = \cancel{I(0)} - y^2 I(a)$

$\frac{d\,I(a)}{I(a)} = -y^2 da$

$\ln(I(a)) = -y^2 a + \cancel{c}^{\,0}$

$I(a) = e^{-y^2 a}$

$\therefore$ $\int y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2} e^{-y^2}$

FIND $\int_0^\infty e^{-\beta x^2 - \alpha/x^2} dx$

LECTURE 4

MORE ON METHODS OF INTEGRATION

I WANT TO POINT OUT THAT LEARNING THE TRICKS OF CALCULUS IS QUITE IMPORTANT BECAUSE WITH A FEW TRICKS YOU CAN SOLVE ALMOST ANY SOLVABLE PROBLEM. BUT YOU WILL NEED THE HELP OF A NUMBER OF KEY INTEGRAL FROM WHICH ALL THE OTHERS ARE DERIVABLE. FOR EXAMPLES IN PIERCE'S TABLE OF INTEGRALS ONLY A COUPLE OF THE DEFINITE INTEGRALS CANNOT BE QUICKLY SOLVED. THE TWO WHICH I HAVE FOUND THAT REQUIRE SPECIAL INGENUITY ARE THE FOLLOWING:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

AND

$$\int_0^{\infty} \frac{x}{\sinh x}\,dx = \frac{\pi^2}{4}$$

IN MOST OF THE DEFINITE INTEGRALS IN PIERCE THEY CAN BE SOLVED BY USING SEVERAL TECHNIQUES SUCH AS INTEGRATION BY PARTS, SUBSTITUTION OF VARIABLES AND DIFFERENTIATING UNDER THE INTEGRAL SIGN. IN THE EXAMPLES ABOUT IT IS OFTEN POSSIBLE TO SOLVE THE INTEGRAL BY SERIES EXPANSION. LET'S WORK THE SECOND INTEGRAL OUT.

WE HAVE THAT $\int_0^{\infty} \frac{x\,dx}{\sinh x}$, SUBSTITUTE FOR $\sinh x$ $\frac{e^{x} - e^{-x}}{2}$

AND REARRANGE

$$\int_0^{\infty} \frac{2e^{-x}\,x\,dx}{1 - e^{-2x}}$$

SINCE WE CAN HANDLE THE INTEGRATION OF e^{-x} THE MORE WE HAVE THE BETTER OFF WE WILL BE. THUS EXPAND THE DENOMINATOR AS

$$2\int_0^{\infty} e^{-x}\,x\,dx\left(1 + e^{-2x} + e^{-4x} + e^{-6x} + \cdots\right) =$$

$$2\int_0^{\infty} x e^{-x}\,dx + 2\int x e^{-3x}\,dx + 2\int x e^{-5x}\,dx + \cdots$$

NOW WE SHOULD REMEMBER THAT

$$\int_0^{\infty} e^{-\alpha x}\,dx = \frac{1}{\alpha} \quad \text{AND} \quad \int_0^{\infty} x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

USING THESE INTEGRALS WE THEN HAVE THAT

$$\int_0^\infty \frac{x\,dx}{\sinh x} = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{9^2} + \cdots\right)$$

NOW I HAPPEN TO KNOW THAT THE SERIES CAN BE SUMMED AND HAS A VALUE OF $\pi^2/8$. IF YOU DON'T KNOW THAT YOU ARE SORT OF STUCK. BUT IF YOU ONLY WANTED THE ANSWER TO A COUPLE OF DECIMAL PLACES YOU WOULD JUST ADD SOME TERMS AS I EXPLAINED BEFORE. THUS WE HAVE THAT

$$\int_0^\infty \frac{x\,dx}{\sinh x} = \frac{\pi^2}{4}$$

I SUGGEST YOU TRY TO WORK OUT THE DEFINITE INTEGRALS GIVEN IN PIERCE (PAGE 62) BECAUSE YOU HAVE ALL THE NECESSARY TOOLS.

SOLVING INTEGRALS USING DIFFERENTIAL EQUATIONS

LAST TIME I WAS TRYING TO WORK OUT THE INTEGRAL

$$S(m) = \int_0^\infty \frac{\cos mx}{1+x^2}\,dx$$

I HAD GOTTEN AS FAR AS WRITING THE INTEGRAL AS A DIFFERENTIAL EQUATION

$$S(m) - S''(m) = \int_0^\infty \cos mx\,dm = \pi\,\delta(m)$$

TO SOLVE THIS EQUATION I CAN FIND A FORMULA IN A BOOK WHICH SAYS THAT THE SOLUTION TO THE EQUATION

$$\frac{d^2y}{dx^2} - y = f(x)$$

IS GIVEN BY

$$y = e^{-x}\int_{-\infty}^{x} e^{t} f(t)\,dt - e^{+x}\int_{x}^{\infty} e^{-t} f(t)\,dt + Ae^{-x} + Be^{+x}$$

WHERE WE HAVE THAT

$$y = S \qquad x = m \quad \text{AND} \quad f = -\pi\delta(m)$$

WE MAY SPECULATE THAT $S(m)$ CONTAINS A DELTA FUNCTION BUT WE SOON REALIZE THAT IF IT DID ITS SECOND DERIVATIVE WOULD BE A HORRIBLE FUNCTION CONTAINING THE SECOND DERIVATIVE OF A DELTA FUNCTION. THUS WE ARE LED TO BELIEVE THAT $S''(m)$ CONTAINS A DELTA FUNCTION. IF THIS WERE THE CASE THE FIRST DERIVATIVE WOULD ~~BE~~ CONTAIN A JUMP OR STEP CHANGE. THUS WE HAVE INTEGRATE OVER A JUMP. THAT MEANS THAT $S(m)$ HAS A KINK IN IT. THE TWO SLOPES DIFFER BY A FACTOR OF π.

IT IS NOW KNOWN THAT IN THE NEIGHBORHOOD OF $m=0$ THE EQUATION FOR $S(m)$ BECOMES

$$S''(m) = -\pi\delta(m)$$

AND

$$S'(m) = -\pi 1(m) + \text{CONSTANT}\begin{cases} = C & m<0 \\ = C-\pi & m>0 \end{cases}$$

AND FINALLY

$$S(m) = -\pi m\, 1(m) + Cm + D$$

IF WE NOTE THAT AS $m \to \infty$ $S(m) \to 0$ WE CONCLUDE $A=0$ AND LIKEWISE AS $m \to -\infty$ $S(m) \to 0$ SO $D=0$. DO TO THE SYMMETRICAL CHARACTER OF $S(m)$ WE HAVE THAT $B=-C$. THUS WE CAN WRITE FOR $m>0$, Be^{-m} AND FOR $m<0$, Be^{+m}. SINCE $S(m) = Be^{-m}$, $m>0$ AND Be^{+m} WE CAN EVALUATE B SINCE

$$S' = \begin{cases} -Be^{-m} & m>0 \\ +Be^{+m} & m<0 \end{cases}$$

$$S'' = \begin{cases} Be^{-m} & m>0 \\ +Be^{+m} & m<0 \end{cases}$$

OR

$$S'' = -2B\,\delta(m)$$

BUT

$$S'' = -\pi\,\delta(m)$$

SO

$$B = \frac{\pi}{2}$$

AND WE HAVE

$$\int_0^\infty \frac{\cos mx}{1+x^2} = S(m) = \frac{\pi}{2}e^{-|m|}$$

QED

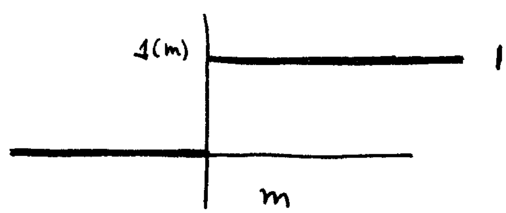
Then our intermediate answer is

$$y = e^{-m}\int_{-\infty}^{m} e^{v}[-\pi\delta(v)]\,dv \;-\; e^{+v}\int_{m}^{\infty} e^{-v}[-\pi\delta(v)]\,dv + Ae^{-m} + Be^{+m}$$

The first integral will give the value $-\pi$ if the integration from - infinity to m contains the delta function $\delta(v)$. If it doesn't then the second integral will give the value $-\pi$. We can write then

$$S(m) = -\pi e^{-m}\,\mathbb{1}(m) + \pi e^{m}(1-\mathbb{1}(m)) + Ae^{-m} + Be^{m}$$

Here the function $\mathbb{1}(m)$ is defined as the step function



$\mathbb{1}(m) = 0 \quad m<0$
$\mathbb{1}(m) = 1 \quad m>0$

Thus

for $m>0$ $\quad S(m) = -\pi e^{-m} + Ae^{-m} + Be^{m}$

And for $m<0$ $\quad S(m) = +\pi e^{m} + Ae^{-m} + Be^{m}$

To go further it is necessary to find A and B. By symmetry of cosine function for m both $\pm$ we have that A and B must be equal. Then if we let $m \to \infty$ the cosine in rapidly oscillatory which damps out eventually so A and B must go to zero.

Now in the range of $m \neq 0$, i.e. where $\delta(m) = 0$ we have the differential equations:

$$S(m) = S''(m) \quad \longrightarrow \quad S(m) = Ae^{m} + Be^{-m} \quad \text{for } m>0$$

And

$$S(m) = S''(m) \quad \longrightarrow \quad S(m) = Ce^{m} + De^{-m} \quad \text{for } m<0$$

We have the problem of fitting these two solutions together at $m=0$. This is commonly called a boundary value problem. Normally these are taught to the student to be quite difficult but the delta function resolves all of the weird conditions developed by the teacher. Both the delta function and step function serve to tell us what happens between the two regions.

ANOTHER INTEGRAL WORTH WORKING OUT IS

$$\int_0^\infty \frac{e^{-ay} - e^{-by}}{y}\, dy$$

WE CAN GET THE ANSWER A NUMBER OF WAYS BUT NOTE IF WE WORK WITH THE FIRST INTEGRAL $\int \frac{e^{-ay}}{y}\, dy$ WE CAN DIFFERENTIATE WRT a AND GET

$$\frac{dI}{da} = -\int_0^\infty e^{-ay}\, dy = -\frac{1}{a}$$

SO

$$I = -\ln a + \text{CONSTANT}$$

TO EVALUATE THE CONSTANT WE NEED TO EVALUATE THE INTEGRAL AT ONE SET OF a AND b VALUES. NOTE IF $a = b$ WE HAVE JUST $\int_0^\infty 0 \frac{dy}{y} = \cancel{\ln y}\Big|_0^\infty = 0$ THEREFORE $I = 0$ AND $C = +\ln b$. THUS

$$I = \ln b/a$$

OBSERVE THAT WE CAN'T TAKE THE LOWER LIMIT OF THE INTEGRAL $\int_0^\infty \frac{e^{-ay}}{y}\, dy$ SINCE THE VALUE BLOWS UP LOGARITHMICALLY IN THE LIMIT. THUS TO GET A FINITE LIMIT LET 0 BE REPLACED BY ϵ AND TAKE THE LIMIT. THEREFORE

$$\int_\epsilon^\infty \frac{e^{-ay}}{y}\, dy = \ln \frac{1.781\, a}{\epsilon} = .5772 - \ln(\epsilon/a)$$

THE NUMBERS ENTERING IN HERE ARE A CURIOUS THING WHICH YOU MAY TRY TO DERIVE.

I WANT TO POINT OUT AND CAUTION YOU NOT TO BE SLOPPY WITH THE CONSTANTS OF INTEGRATION. IN THE CASE WHERE I SPLIT OF THE INTEGRAL INTO

$$\int_0^\infty \frac{e^{-ay}}{y}\, dy - \int_0^\infty \frac{e^{-by}}{y}\, dy$$

THE INTEGRATION GIVES

$$-\ln a + \infty + \ln b - \infty$$

AND YOU CAN CASUALLY SUBTRACT OUT THE TWO INFINITIES TO GIVE 0. THIS IS BAD AND VERY ILLEGITIMATE.

To show you what can happen suppose I substitute for y, z/a. Then I have

$$\int_0^\infty \frac{e^{-z}\,dz/a}{z/a} - \int_0^\infty \frac{e^{-z/a}\,dz}{z} = ?$$

Whenever you transform limits which go to ∞ you can get in trouble. The best bet is to put ϵ and keep the limits finite; then you can combine the answer so it does not depend on ϵ. In the above case where we set $z = ay$ let

$$\int_{\epsilon a}^\infty \frac{e^{-z}dz}{z} - \int_{\epsilon b}^\infty \frac{e^{-z}dz}{z} = \int_{\epsilon a}^{\epsilon b} \frac{e^{-z}dz}{z} \approx \int_{\epsilon a}^{\epsilon b} \frac{1\,dz}{z}$$

$$= \ln\left(\frac{\epsilon b}{\epsilon a}\right) = \ln(b/a)$$

As a rule when I go through a problem the first time I and usually quite sloppy, not paying attention to signs, pi's, factors of 2 etc. The reason is that more often than not my first approach leads to a dead end so I have to retreat to another tact. If I see I finally have a way to the answer I then go back very carefully putting in all the correct factors, and grind-grind-grind.

I mentioned that when an integral is solvable there are a 100 different ways to work it out. To show you what I mean consider again the integrate $\int_0^\infty \frac{\cos mx}{1+x^2}\,dx$. This time will kill the problem with a sledge hammer. Suppose we knew that

$$\int_0^\infty e^{-\alpha x}\cos\beta x\,dx = \frac{\alpha}{\alpha^2+\beta^2}$$

Then I can substitute an integral into the integral for a piece which is hard to handle. Hopefully the resulting integral is easier to solve. Thus

$$\int_0^\infty e^{-t}\cos xt\,dt = \frac{1}{1+x^2}$$

WE THEN HAVE THAT

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \int_0^\infty dx \int_0^\infty dt\ e^{-x} \cos xt \cos mx\ dt$$

NOW DON'T BE AN IDIOT AN INTEGRATE WRT t FIRST BECAUSE YOU WILL BE RIGHT BACK WHERE YOU STARTED FROM. SUBSTITUTE for

$$\cos xt \cos mx = \tfrac{1}{2}\left[\cos(m+t)x + \cos(m-t)x\right]$$

AND RECALL

$$\int_0^\infty \cos Kx\, dx = \pi\delta(K)$$

THEN

$$\int_0^\infty e^{-t}\ \frac{\pi}{2}\left[\delta(m+t) + \delta(m-t)\right] dt$$

YOU HAVE TO BE CAREFUL HERE IF WE TAKE m TO BE GREATER THAN 0 THEN WE TAKE $\delta(m-t)$ AND GET

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m} \qquad m>0$$

WE CAN TRY ANOTHER METHOD USING SERIES EXPANSION, I.E

$$\int \frac{\cos mx}{1+x^2}\, dx = \int \cos mx\left(1 - x^2 + x^4 - x^6 + x^8 + \cdots\right) dx$$

$$= \pi\delta(m) + \pi\delta''(m) + \pi\delta''''(m) + \cdots$$

NOW THIS IS A MESS AND I'D STOP AND TRY ANOTHER METHOD. HOWEVER YOU COULD PRESS ON USING THE DIFFERENTIAL OPERATOR NOTATION

$$\int \frac{\cos mx}{1+x^2}\, dx = \pi\left[1 + D^2 + D^4 + D^6 + \cdots\right] = \frac{1}{1-D^2}\,\pi\delta(m)$$

NOW YOU HAVE TO SOLVE

$$(1-D^2)\,S(m) = \pi\delta(m)$$

or

$$S(m) - S''(m) = \pi\delta(m)$$

AND YOU'RE RIGHT BACK WHERE YOU STARTED FROM.

PROBLEM - LECTURE 4

Show

$$\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$$

$$\int_0^{\pi/2} \ln(\sin x)\, dx = \frac{\pi}{2} \ln \tfrac{1}{2}$$

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = \frac{\pi}{8} \ln 2$$

$$\int_0^{\pi/2} \frac{x \sin x}{1+\cos^2 x} = \frac{\pi^2}{4}$$

$$\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}}\left[\frac{\pi}{2} + \ln(\sqrt{2}-1)\right]$$

$$\int_0^\infty \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin a\pi}$$

$$\int_0^\infty \frac{\sinh ax}{\sinh x}\, dx = \frac{1}{2}\tan a/2$$

LECTURE 5

SOLVING DEFINITE INTEGRALS

I'd like to go over some of the problems I assigned last time so you can get a feel how I go about solving these definite integrals

Suppose I want the integral

$$I = \int_0^\infty \frac{dx}{x^4+a^4}$$

To solve this I might try factoring since I know that

$$\int_0^\infty \frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}$$

I have that

$$I = \int \frac{dx}{(x^2+ia^2)(x^2-ia^2)} = \frac{1}{2ia^2}\int \left[\frac{1}{x^2+ia^2} - \frac{1}{x^2-ia^2}\right] dx$$

$$I = \frac{\pi}{2ia^2}\left[\frac{1}{a\sqrt{i}} - \frac{1}{a\sqrt{-i}}\right] = \frac{\pi}{2ia^3}\left[\frac{\sqrt{-i}-\sqrt{i}}{\sqrt{i}\sqrt{-i}}\right]$$

Now I have to have the square root of plus and minus i. To get $\sqrt{i}$ and $\sqrt{-i}$ I take a complex circle and get the hypotenuse as

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$$

$$\sqrt{-i} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1-i)$$

Then I have

$$I = \frac{\pi}{2\sqrt{2}a^3} = \frac{\sqrt{2}\pi}{4a^3}$$

Note that the i's cancelled out so I have the right roots of $\sqrt{i}$ and $\sqrt{-i}$. If I had taken the negative roots I would have gotten the wrong answer - it would have been imaginery.

SUPPOSE I WANTED THE VALUE OF THE FOLLOWING INTEGRAL

$$F(m) = \int_0^\infty \frac{\cos mx}{1+x^4}\, dx$$

NOTE IF DIFFERENTIATE F(m) FOUR TIMES WRT m AND ADD F(m) I G GET

$$F^{IV}(m) + F(m) = \int_0^\infty \cos mx \, dx = \pi \delta(m)$$

THE FUNCTION f(m) IS SYMMETRICAL SINCE THE 4TH DERIVATIVE OF F EQUALS THE NEGATIVE OF ITSELF. A FUNCTION WHICH HAS THIS CHARACTER IS

$$F(m) = e^{\alpha m}$$

THEN THE DIFFERENTIAL EQUATION GIVES AS THE ROOTS $\alpha^4 = -1$. THIS EQUATION HAS FOUR ROOTS ARISING FROM THE QUADRATIC $\alpha^2 = \pm i$:

$$\alpha_1 = \frac{1}{\sqrt{2}}(1+i) \qquad \alpha_2 = \frac{1}{\sqrt{2}}(1-i) \qquad \alpha_3 = \frac{1}{\sqrt{2}}(i-1) \qquad \alpha_4 = \frac{-1}{\sqrt{2}}(i+1)$$

THE FORM OF THE SOLUTION IS THEN

$$F(m) = A e^{\alpha_1 m} + B e^{\alpha_2 m} + C e^{\alpha_3 m} + D e^{\alpha_4 m}$$

WE NOW OBSERVED THAT AS $m \to \infty$ F(m) SHOULD APPROACH 0; THEREFORE A AND B = 0 OTHERWISE THE FUNCTION EXPLODES. NOW WE HAVE THAT

$$F(m) = C e^{-\frac{m}{\sqrt{2}} - \frac{im}{\sqrt{2}}} + D e^{-m/\sqrt{2} + im/\sqrt{2}}$$

WHICH CAN BE REWRITTEN AS

$$F(m) = e^{-m/\sqrt{2}} \left[C' \cos m/\sqrt{2} + D' \sin m/\sqrt{2} \right]$$

WHERE C' AND D' HAVE TO BE REAL WHERE C AND D COULD BE COMPLEX. WE NOW HAVE TO FIND 2 REAL CONSTANTS. WHEN $m < 0$ WE HAVE THAT $F(m) = e^{+m/\sqrt{2}} (C' \cos m/\sqrt{2} - D' \sin m/\sqrt{2})$. THE DELTA FUNCTION MUST RELATED THESE TWO VALUES OF F(m). WE KNOW THAT THE 4TH DERIVATIVE CONTAINS A DELTA FUNCTION THUS THE DERIVATIVES GIVES US FOUR EQUATION TO FIND 2 UNKNOWN:

$f(m) = f(-m)$, $f'(m) = F'(-m)$, $f''(m) = F''(-m)$

$f'''(+0) - F'''(-0) = \pi$, THE JUMP GIVES US THE δ-FUNCTION IN THE 4TH DERIVATIVE.

LET'S TRY TO FIND THE INTEGRAL

$$I = \int_0^{\pi/2} \ln(\sin x)\, dx$$

FIRST TRY INTEGRATING BY PARTS,

$$\int_0^{\frac{\pi}{2}} x \frac{\cos x}{\sin x}\, dx = \int x \cot x\, dx$$

NOW I'M STUCK. HOW DO I INTEGRATE $x \cot x$? I DON'T KNOW. WELL I TRY AGAIN. I COULD EXPAND BY SERIES AND INTEGRATE TERM BY TERM. THAT'S A HELLUVA WAY TO RUN A RAILROAD. SUPPOSE I MAKE THE FOLLOWING OBSERVATION THAT

$$\int_0^{\pi/2} \ln(\sin x)\, dx = \int_0^{\pi/2} \ln(\cos x)\, dx = A$$

THEN
$$2A = \int_0^{\pi/2} [\ln(\sin x) + \ln(\cos x)]\, dx$$

COMBINE
$$= \int_0^{\pi/2} \ln(\sin x \cos x)\, dx = \int_0^{\pi/2} [\ln(\tfrac{1}{2} \sin 2x)]\, dx$$

AND
$$2A = \int_0^{\frac{\pi}{2}} \ln 1/2\, dx + \int_0^{\pi/2} \ln(\sin 2x)\, dx$$
$$= -\frac{\pi}{2} \ln 2 + \int_0^{\pi/2} \ln(\sin 2x)\, dx$$

IF I NOW SUBSTITUTE $Y = 2X$ I GET

$$2A = -\frac{\pi}{2} \ln 2 + \int_0^{\pi} \ln(\sin y) \frac{dy}{2}$$

NOW THE INTEGRAL IS JUST TWICE THE ONE I STARTED WITH SO

$$2A = -\frac{\pi}{2} \ln 2 + \frac{1}{2}(2A) \longrightarrow A = -\frac{\pi}{2} \ln 2$$

OR
$$\int_0^{\pi/2} \ln(\sin x)\, dx = \int_0^{\pi/2} \ln(\cos x)\, dx = -\frac{\pi}{2} \ln 2$$

NOW LET'S TRY TO INTEGRATE

$$I = \int_0^1 \frac{\ln(1+x)}{(1+x^2)}\, dx$$

LET'S SUBSTITUTE $x = \tan y$ SINCE I KNOW THAT

$$dx = \frac{dy}{\cos^2 y} \Rightarrow \frac{dx}{1+x^2} = dy$$

THEN

$$I = \int_0^{\pi/4} \ln(1+\tan y)\, dy$$

$$= \int_0^{\pi/4} \ln(\cos y + \sin y)\, dy - \int_0^{\pi} \ln(\cos y)\, dy$$

NOW I'M WORRIED ABOUT THE INTEGRATION LIMIT $\pi/4$. SO I ~~real~~ remember THE TRIGONOMETRIC IDENTITY

$$\cos y + \sin y = \sqrt{2}\cos(y - \pi/4)$$

I.E., TWO SINE WAVES CAN BE COMBINED INTO ONE SINE WAVE.

$$I = \int_0^{\pi/4} \ln\sqrt{2}\, dy + \int_0^{\pi/4} \ln\left[\cos\left(\tfrac{\pi}{4} - y\right)\right] dy - \int_0^{\pi/4} \ln\cos y\, dy$$

IF $z = \pi/4 - y$ THEN THE LAST TWO INTEGRALS SUBTRACT OUT. AND WE HAVE

$$I = \frac{\pi}{4}\ln\sqrt{2} = \frac{\pi}{8}\ln 2$$

LET ME INTEGRATE $\displaystyle\int_0^{\pi} \frac{x\sin x}{(1+\cos^2 x)}\, dx = I$

$$I = \int_0^{\pi/2} \frac{x\sin x}{1+\cos^2 x}\, dx + \underbrace{\int_{\pi/2}^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx}_{\int_0^{\pi/2} \frac{(\pi-y)\sin y}{1+\cos^2 y}\, dy}$$

THEN

$$I = \int_0^{\pi} \frac{\pi \sin y\, dy}{1+\cos^2 y} = \frac{\pi^2}{4}$$

NOW WE'LL TRY $\int_0^\infty \frac{e^{ax}}{1+e^x}\,dx$

EXPAND BY SERIES,

$$I = \int_0^\infty e^{ax}\left(e^{-x} - e^{-2x} + e^{-3x} - e^{-4x} + \cdots\right) dx$$

$$= \frac{1}{1-a} - \frac{1}{2-a} + \frac{1}{3-a} - \frac{1}{4-a} = S(a)$$

THIS LAST SERIES HAS TO BE SUMMED AND IT IS NOT EASY TO DO. THE ANSWER IS

$$S(a) = \frac{\pi}{\sin \pi a}$$

I'LL HAVE TO COME BACK AND SHOW YOU HOW TO SUM THIS SERIES.

THIS PROBLEM IS of THE SAME CLASS AS

$$\int_0^\infty \frac{\sin ax}{\sinh x}\,dx$$

NOTE IF I SUBSTITUTE $\sinh x = \frac{e^x - e^{-x}}{2}$

$$I = \int 2 \sin ax \left(\frac{e^{-x}}{1 - e^{-2x}}\right) dx$$

$$= 2\int_0^\infty \sin ax\left(e^{-x} + e^{-3x} + e^{-5x} + \cdots\right)$$

AND NOW I USE THE FACT THAT

$$\int_0^\infty \sin \beta x\, e^{-px}\,dx = \frac{\beta}{\beta^2 + p^2}$$

I GET THEN

$$S(a) = 2a\left[\frac{1}{1+a^2} + \frac{1}{9+a^2} + \frac{1}{25+a^2} + \cdots\right]$$

$$= 2a \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2+a^2}$$

LECTURE 5

PROBLEMS

$$\int \frac{dx}{1+x^4}$$

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-a})$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^2}$$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

$$\int_0^\infty \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}$$

$$\int_0^\infty \frac{dx}{(x^2+b^2)(x^2+a^2)^2} = \frac{\pi(2a+b)}{2a^3b(a+b^3)}$$

$$\int_0^\infty \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi}{4}(1+m)e^{-m} \qquad m>0$$

$$\int_0^\infty \frac{\sin nx}{(\sinh x)}\frac{dx}{(1+x^2)} = \frac{\pi(e^n-1)}{(e^2-1)e^{n-1}}$$

$$\int_0^\pi \ln(c^2-2ac\cos\theta+a^2)\,dx = \begin{cases} 2\pi\ln a & \text{if } a>c \\ 2\pi\ln c & \text{if } a<c \end{cases}$$

$$\int_0^\pi \frac{\cos 2x}{(1-2a\cos x+a^2)}\,dx = \frac{\pi a^2}{1-a^2}$$

COMPLEX NUMBERS

Complex numbers are centered around one fundamental equation

$$X^2 = -1$$

In order to solve this equation it was necessary to invent the symbol i which is defined as having the property that $i^2 = -1$. With this definition i then amazingly enough to satisfy all the basic algebraic relationships. For example multiplication would give

$$(i + i)^2 = (2i)^2 = 4(i^2) = -4$$

or

$$= i^2 + 2ii + i^2 = -1 - 2 - 1 = -4$$

Thus when I have a complex number made up of an imaginery and real part such as $a + bi$ where a and b are real I can multiply as follows,

$$(5 + 3i)(7 + 2i) = 35 + 31i - 6 = 29 + 31i$$

Multiplying and adding complex numbers gives a new complex number.

Dividing two complex numbers involves taking the complex conjugate of the denominator and multiplying top and bottom. E.G

$$\frac{7+2i}{5+3i} = \frac{7+2i}{5+3i}\,\frac{(5-3i)}{(5-3i)} = \frac{(7+2i)(5-3i)}{34} = \frac{41+4i}{34}$$

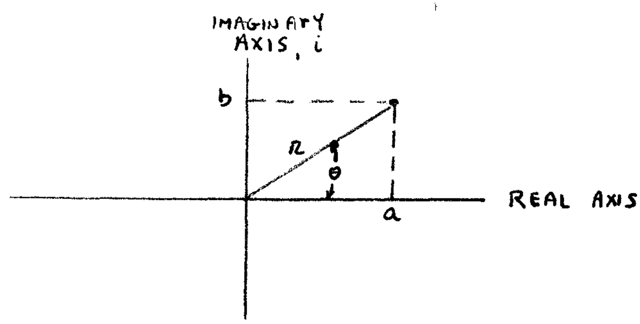
The complex conjugate is obtained by changing all the signs on i.

Suppose we wanted $X^2 = i$ solved. To do this let $X = a + bi$ then I have $(a+bi)^2 = i$ and $a^2 + 2abi - b^2 = i$ which gives two equations $a^2 - b^2 = 0$ and $2ab = 1$. One root is $a = b$ then $a = 1/\sqrt{2} = b$. Thus $X = \frac{1}{\sqrt{2}}(1 + i)$. Since $a = -b$ is also a root X is + or -.

The amazing thing about complex numbers that once we defined i we never have to make another similar definition. We have opened the door to a whole new world.

LECTURE 6
COMPLEX NUMBERS

LAST TIME WE STARTED TO TALK ABOUT COMPLEX NUMBERS. IT IS CONVENIENT WHEN WORKING WITH COMPLEX NUMBERS (C.N.) TO DRAW A DIAGRAM. THE PLANE THEN REPRESENTS THE REAL AND IMAGINARY PART OF THE C.N. IN THE DIAGRAM THAT FOLLOWS WE HAVE REPRESENTED THE GENERAL C.N. $a + bi$.

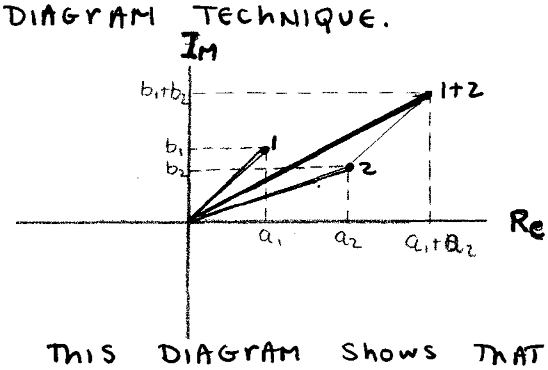


ANY C.N. THEN CAN BE REPRESENTED AS A POINT IN THE COMPLEX PLANE AND LOCATED BY THE COORDINATES a AND b. IT IS OFTEN TIME CONVXENIENT TO REPRESENT THE C.N. IN POLAR COORDINATE NOTATION. IF WE LET

$$a = r\cos\theta \quad \text{AND} \quad b = r\sin\theta$$

WHERE $\theta = \tan^{-1} b/a$ AND $r = \sqrt{a^2+b^2}$ WE HAVE LOCATED THE SAME POINT BY AN ANGLE θ FROM THE REAL AXIS AND A LENGTH, r, FROM THE ORIGIN.

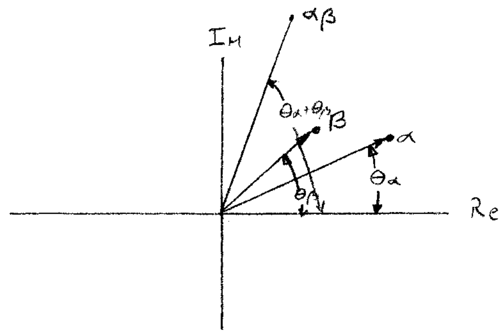
WHEN TWO C.N.S ARE ADDED WE CAN DESCRIBE THE GEOMETRICAL SIGNIFICANCE OF THIS OPERATION THROUGH THE DIAGRAM TECHNIQUE.



THIS DIAGRAM SHOWS THAT THE TWO C.N'S ADD JUST LIKE VECTORS ACCORDING TO THE WELLKNOWN PARALLELOGRAM RULE.

MULTIPLICATION of COMPLEX NUMBERS

When two C.N.'s are multiplied together the resulting product is more difficult to see even diagrammatically. Suppose I have two C.N.'s $\alpha = a + ib$ and $\beta = c + id$. If I multiply them together two things happen - the product is stretched and rotated. Using the diagram we have,



Let's work the product out using polar coordinates.

$$\alpha = |\alpha| \, (\cos\theta_\alpha + i \sin\theta_\alpha) \quad \text{And} \quad \beta = |\beta| \, (\cos\theta_\beta + i \sin\theta_\beta)$$

The product is

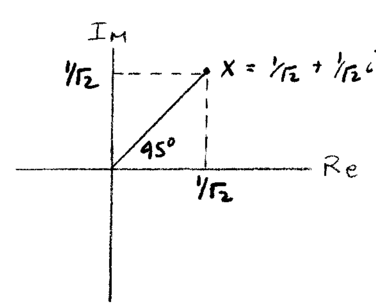
$$\alpha\beta = |\alpha\beta| \left[\cos\theta_\beta \cos\theta_\alpha - \sin\theta_\alpha \sin\theta_\beta + i\left(\sin\theta_\beta \cos\theta_\alpha + \sin\theta_\alpha \cos\theta_\beta\right)\right]$$

Using the trig. identity we have

$$\alpha\beta = |\alpha\beta| \left[\cos(\theta_\alpha + \theta_\beta) + i \sin(\theta_\alpha + \theta_\beta)\right]$$

Therefore the new length is $|\alpha\beta|$ and the angle is the sum of the two original angles. You should notice if I have a C.N. and multiply it by i the number or vector is rotated 90 degrees. Multiplying a C.N. by -1 simply changes its value by $180°$.

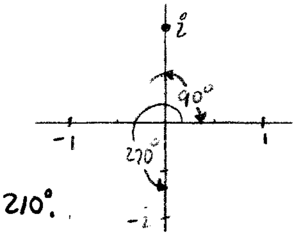
Now we find $\sqrt{i}$ using the geometrical properties of C.N's just described. Let $\sqrt{i} = x$ Then $x^2 = i$. The C.N. x is then a number which has unit length Thus $\sqrt{a^2+b^2} = 1$ And $a = b$ so $a = 1/\sqrt{2}$ And $x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$. Of course we have the negative root which lies in the other quadrant. The clue that the result is $1/\sqrt{2} + 1/\sqrt{2}\, i$ is the fact that we need an angle which when doubled gives $90°$; This is of course $45°$.



Now that we are getting smart at working with C.N's let try to solve

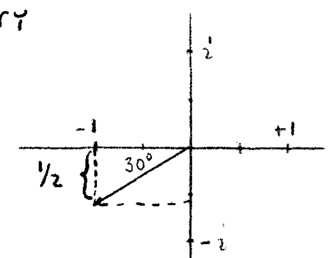
$$\sqrt[3]{-i} = ?$$

If we cube both sides then we have $X^3 = -i$. Now we need a number when rotated through 3 equal angles gets us to $-i$ which is at $\theta = 270°$. Thus $\theta/3 = 90°$ which means one root is i itself. Now there are two other roots which we have to find. How about adding 360° to 270°; surely I end up where I started but now when I divide by 3 again I get $630/3 = 210°$. Thus another root is at a θ angle of 210°.



To find its components draw a diagram: The length must unity so the imaginary part is $-\sin 30° (1) = -1/2$. This makes the real part $\sqrt{1-1/4} = -\sqrt{3}/2$. Thus we have another root

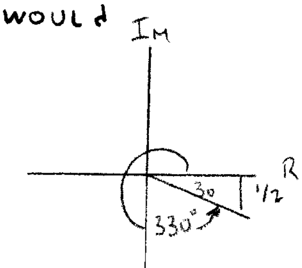
$$X_2 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$



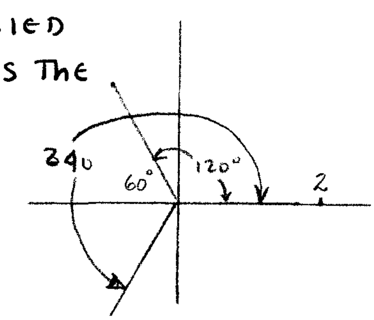
To check this answer cube it and see if you get $-i$ — you will! We still have another root to find so let's add another 360° to 630°. We get 990° Now dividing by 3 again we get 330°. Thus the other root ~~negati~~ would be

$$X_3 = -\sqrt{3}/2 + i/2$$

Thus we have found the 3 cube roots of $-i$.



In general every number has 3 cube roots. Consider, 8, it has 2 as one cube root. To find the other 2, notice the length of the C.N. must be 2 and ~~an~~ when its angle mutliplied by 3 we get back to 0°. Thus $360°/3 = 120°$ is the position of one other root. The components are -1 and $i/\sqrt{3}$. To find the 3rd root we add anothe 360 divide by 3 and get 240°. The other root is $-1 - i/\sqrt{3}$. If we add 360° again we get 2 again so that's all the roots.



As a general root it is possible to take any integral root of a complex or real number. The roots will lie on radii located by dividing 360° into n equal parts, like a piece of pie. E.G to find $a^{1/6}$ the roots are located every 360/6 or 60° around the unity circle. They form the spokes of a hexagon.

If one the other hand you want the π^{th} root of a number, you are out of luck because there are an infinite number of them.

POWERS OF COMPLEX NUMBERS

In order to work out a problem like 2^{1+i} we will need to know the product of $\cos\theta + i\sin\theta$ and $\cos\varphi + i\sin\varphi$. If we let $R^\theta = \cos\theta + i\sin\theta$ and $R^\varphi = \cos\varphi + i\sin\varphi$, then wouldn't it be nice if the product were just $R^\theta \cdot R^\varphi = R^{\theta+\varphi}$.

To see if we can prove such a theorem let's define R to be $\cos 1 + i\sin 1$ which is a perfectly good complex number, i.e. $R = .540 + i(.841)$. Now R^2 would equal $\cos 2 + i\sin 2$. And in general it holds that for any real n $R^n = \cos n + i\sin n$. Now if the theorem holds and $R^\theta R^\varphi = R^{\theta+\varphi}$ the number R must be some sort of fundamental root from which all numbers can be taken. Indeed this turns out to be the case.

What I want to prove is that $R^\theta = (e^i)^\theta = e^{i\theta}$. To do that consider $(R^{\theta/N})^N = R^\theta = (\cos\theta/N + i\sin\theta/N)^N$ and let N be very large, E.G $10^6, 10^9, 10^\infty$. If we write the sine, cosine series inside the parentheses then the leading terms are $1 + i\theta/N - \theta^2/2N^2 + \ldots$. The $1/N^2$ terms will drop out as N gets large. We have then to take the limit as $N \to \infty$ of $(1 + i\theta/N)^N$. This has been established to be

$$\lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N \equiv e^x$$

Thus if $x = i\theta$

$$\lim_{N\to\infty} (+ i\theta/N) = e^{i\theta} = R^\theta$$

To answer the question another way if $R^{\theta} = (F^{i})^{\theta}$ what is F? We know that $R^{\theta} R^{-\theta} = 1$ by the rule for taking complex conjugates. Then it must be true that

$$(F^{i\theta})\left((F^{*})^{-i\theta}\right) = 1$$

or

$$\left(\frac{F}{F^{*}}\right)^{i\theta} = 1$$

Thus for this to hold $F = F^{*}$ so F must be real. It turns out that $F = 2.72\ldots$.

The fact that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

is one of the most remarkable equations in all of mathematics - it is not *the* most remarkable equation. It is this equation that turns a boy into a man; a student into a mathematician. It is ~~the~~ a very delightful thing. But the most remarkable formula in all of mathematics (by a somewhat subjective evaluation) is

$$\boxed{e^{i\pi} + 1 = 0}$$

Feynman loved this equation

This equation contains all the essential actions of mathematics plus all the key symbols, 1, 0, e, i, π, and the equality sign. The operations involved are addition multiplication and taking exponenentials. It involves not only real numbers but imaginary number. In English this says that $e^{i\pi} = -1$. This is a curious fact because it can be interperted as saying

$$(e^{\pi})^{i} = -1 \qquad \text{or} \qquad (23)^{i} \approx -1$$

Similarly $e^{2\pi i} = +1$ or that $(546)^{i} \approx +1$. Thus the i^{th} ~~roo~~ power of irrational numbers can produce real numbers - a most curious fact of complex numbers! I should point out that $\sqrt[i]{1} \approx 546$!

Why would the ith root of 1 produce 546?? Maybe this is the right answer instead of "42" as suggested in the Hitchhiker's Guide to the Galaxy!

The general formula for taking the complex power of a complex number is

$$z_1^{z_2} \quad \text{where } z_1 = a+bi \text{ and } z_2 = c+di$$

If $z_1 = re^{i\theta}$ then $(re^{i\theta})^{c+di} = r^{c+di} e^{ci\theta - d\theta}$ which simplies to

$$r^c e^{ci\theta} r^{di} e^{-\theta d}$$

Now this formula is almost understandable except what does r^{di} mean? r is just a length therefore I have $r^{di} = (r^d)^i =$ (real number) raised to i^{th} power. If I let $r^d = e^t$ where $t \equiv d \ln r$ then $(e^t)^i = e^{iT} = \cos t + i \sin t$.
As an example find i^{-2i}. According to the formula

$$i^{-2i} = e^{-2i \ln i} = e^{-2i[\ln 1 + i(\pi/2 + 2n\pi)]}$$
$$= e^{-2i[0 + i(\pi/2 + 2n\pi)]} = e^{\pi + 4n\pi}$$

Logarithms of complex number

We next want to consider taking the logarithm of a complex number, i.e., $\ln(a+bi)$. To find out what this is let's define

$$\ln(a+bi) = x + iY$$

Then this is equivalent to

$$a + ib = e^{x+iY} = e^x e^{iY} = e^x(\cos Y + i \sin Y)$$

Now we have two equations to solve

$$a = e^x \cos Y$$
$$b = e^x \sin Y$$

From which we obtain $\sqrt{a^2+b^2} = e^x$, $X = \ln(\sqrt{a^2+b^2})$ and $Y = b/a$. Thus we have

$$\ln(a+bi) = \ln\sqrt{a^2+b^2} + i \tan^{-} b/a$$

or in polar notation

$$\ln(re^{i\theta}) = \ln r + i\theta$$

Since θ can be any value (each differing by 2π), a complex number has infinitely many logarithms differing from each other by multiples of $2\pi i$.

ON EXTRA TREAT OF TAKING THE LOG OF COMPLEX NUMBERS IS THAT WE CAN NOW TAKE THE LOG OF A NEGATIVE NUMBER, E.G

$$\ln(-1) = i\pi$$

THE $\ln$ OF i CAN BE FOUND SIMPLY SINCE

$$\ln(i) = \ln\sqrt{0^2+1^2} + i\tan^{-1} 1/0 = \ln 0 + i\tan^{-1}\infty =$$
$$= i\pi/2$$

TRIGONOMETRIC FUNCTIONS OF COMPLEX NUMBERS

LET'S PURSUE THE IDEA OF C.N'S FURTHER AND SEE IF WE CAN GIVE MEANING TO THE SYMBOLS $\sin(2+3i)$ OR GENERALLY $\sin z$ WHERE z IS ANY COMPLEX NUMBER. SUPPOSE WE JUST TOOK FOR FAITH THAT

$$e^{iz} = \cos z + i\sin z$$

WHERE z MAY BE COMPLEX OR REAL. THEN WE CAN SOLVE FOR $\sin z$ AND $\cos z$ TO GET

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{AND} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

THE INVERSE TRIG FUNCTIONS CAN ALSO BE DEFINED IF

$$w = \cos z \quad \text{THE} \quad z = \cos^{-1} w$$

LIKEWISE FOR THE SINE. SUPPOSE WE WANT THE INVERSE TANGENT, I.E.

$$w = \tan^{-1} z$$

WHICH IMPLIES

$$z = \tan w$$

SUBSTITUTING FOR TANGENT

$$z = \frac{(e^{iw} - e^{-iw})}{i(e^{iw} + e^{-iw})}$$

COLLECTING TERMS

$$iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \longrightarrow iz(e^{2iw} + 1) = e^{2iw} - 1 \longrightarrow \frac{1+iz}{1-iz} = e^{2iw}$$

THEN TAKING LOGS $$w = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) = \tan^{-1} z$$

EXPONENTIALS ARE REALLY SINES AND COSINES AND IN THE COMPLEX NUMBER SYSTEM THEY ARE TIED CLOSELY TOGETHER.

LET'S GO BACK AND TAKE A CLOSER LOOK AT $\cos(x+iy)$. EXPANDING OUT WE GET

$$\cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{-y}e^{ix} + e^{+y}e^{-ix}}{2}$$

$$= \frac{1}{2}\left[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right]$$

$$= \left(\frac{e^{y} + e^{-y}}{2}\right)\cos x + i\left(\frac{e^{y}-e^{-y}}{2}\right)\sin x$$

IT IS SOMETIMES CONVENIENT TO DEFINE THE NEW FUNCTIONS

$$\sinh y = \frac{e^{y}-e^{-y}}{2} \quad \text{AND} \quad \cosh y = \frac{e^{y}+e^{-y}}{2}$$

which ARE CALLED hyPERbolic FUNCTIONS. THEY CAN ALTERNATELY BE defined AS

$$\cosh y = \cos(iy) \quad , \quad \sinh y = -i\sin iy$$

THE $\cosh y$ AND $\sinh y$ HAVE THE PROPERTY THAT

$$\cosh^2 y - \sinh^2 y = 1$$

PROBLEM: FIND $(1+i)^{1-i} = z$

This is of the form

$$(a+bi)^{c+di} = (r e^{i\theta})^{c+di}$$

where $a = 1 \quad b = 1 \quad c = 1 \quad d = -1 \qquad r = \sqrt{2} \quad \theta = \pi/4$

$$z = \left(\sqrt{2}\, e^{i\pi/4}\right)^{c+di} = (\sqrt{2}\, e^{i\pi/4})^{c} \; (\sqrt{2}\, e^{i\pi/4})^{di}$$

$$= \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right) \left(\sqrt{2}^{\,di}\, e^{-d\pi/4}\right)$$

$$= \sqrt{2}\,(e^{+\pi/4 + 2\pi n}) \, (\cos\pi/4 + i \sin \pi/4) \, (\sqrt{2})^{di}$$

Now $(\sqrt{2})^{di} = r^{di} = \cos t + i \sin t \quad$ where $t = d \ln r$

$t = -1 \ln\sqrt{2} \quad$ so that $\quad \sqrt{2}^{-i} = \cos(-\ln\sqrt{2}) + i \sin(-\ln\sqrt{2})$

$$z = \sqrt{2}\, e^{+\pi/4 + 2n\pi} \left[\cos\pi/4 + \cos(-\ln\sqrt{2}) + i\left(\sin\pi/4 + \sin(-\ln\sqrt{2})\right)\right]$$

$$= \sqrt{2}\, e^{\pi/4 + 2n\pi} \left[\cos\left(\pi/4 - \ln\sqrt{2}\right) + i \sin\left(\pi/4 - \ln\sqrt{2}\right) \right]$$

CONTOUR INTEGRATION

FUNCTIONS OF A COMPLEX VARIABLE

I NOW LIKE TO TALK ABOUT A MORE GENERAL SUBJECT THAT OF THE PROPERTIES OF FUNCTIONS WHICH CONTAIN COMPLEX VARIABLES. I WILL DEAL WITH FUNCTIONS OF THE TYPE $w = f(z)$ WHERE z AND w ARE BOTH COMPLEX NUMBERS (C.N.) AN EXAMPLE MIGHT BE

$$w = z^2 \qquad z = x + iy$$

THEN $w = u + iv$ IS DESCRIBABLE IN TERMS OF x AND y SINCE $z^2 = (x^2 - y^2) + 2ixy$. THEN WE HAVE THAT

$$u = x^2 + y^2 \quad \text{AND} \quad v = 2xy$$

ANOTHER EXAMPLE OF $w = f(z)$ IS

$$w = \frac{1}{1 + iz}$$

AND TO FIND u AND v I HAVE TO REWRITE w AS

$$w = \frac{1}{1 + i(x + iy)} = \frac{1}{(1-y) + ix} = \frac{(1-y) - ix}{(1-y)^2 + x^2}$$

OR

$$u = \frac{1-y}{(1-y^2) + x^2} \quad \text{AND} \quad v = \frac{-x}{(1-y)^2 + x^2}$$

YOU SEE THEN THAT A COMPLEX NUMBER OF A COMPLEX FUNCTION IS A FUNCTION OF 2 REAL VARIABLES.

IT IS HARD TO PLOT w BECAUSE FOR EVERY z WHICH REQUIRES 2 REAL VARIABLES TO LOCATE IT THERE ARE 2 REAL VARIABLES IN THE w PLANE. THUS WE HAVE A FOUR DIMENSIONAL MAPPING.

ANOTHER EXAMPLE OF $w = f(z)$ IS THE FUNCTION $w = z^*$ WHERE WE WANT THE COMPLEX CONJUGATE OF z.

THUS
$$u + iv = (x + iy)^* = x - iy$$
$$u = x \quad \text{AND} \quad v = -y$$

ONE MORE EXAMPLE: $w = e^z = e^{x+iy}$

$$w = e^x(\cos y + i \sin y)$$
$$u = e^x \cos y \qquad v = e^x \sin y$$

So far we haven't said too much about the general properties of a complex function. There is a difference between certain functions and the difference has to do with whether the function is analytic or not. An analytic function has the property that it can be differentiated. That means that u and v are not arbitrary and have a special property that they come from the same function.

Some functions are not analytic. For instance what is the derivative of z^*? where $*$ denotes taking the complex conjugate. How do you differentiate an operation like that? Something must be the matter. Let's define the derivative as

$$F'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

And now try the function z^*,

$$g(z) = z^* = (x+iy)^* = (x-iy)$$

Then

$$g'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^* - z^*}{\Delta z}$$

where $\Delta z = \Delta x - i\Delta y$. Thus

$$g'(z) = \lim_{\Delta z \to 0} \frac{(\Delta z)^*}{\Delta z} = \lim_{\Delta x + i\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

This limit depends on the ratio of $\Delta x/\Delta y$. It is either $+1$ or -1. Therefore the limit depends on how the 0 is approached. An analytic function is defined in the limit as $\Delta z \to 0$ from any direction, eg. if $w = z^2$ the $f' = 2z$ or

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim \frac{2z\Delta z}{\Delta z} = 2z$$

In order for a function to be analytic it must satisfy a criterion called the Cauchy - Rieman criteria. Let's derive the criteria for $w = f(z)$ where the real and imaginery parts of w are given by $u = u(x,y)$ and $v = v(x,y)$ respectively.

EXPANDING OUT THE DERIVATIVE

$$f'(z) = \lim_{\Delta x + i\Delta y \to 0} \frac{W(z+\Delta z) - W(z)}{\Delta z}$$

$$= \lim \frac{[U(x+\Delta x, y+\Delta y) + iV(x+\Delta x, y+\Delta y)] - [U(x,y) + iV(x,y)]}{\Delta x + i\Delta y}$$

$$= \frac{\Delta x \, \partial U/\partial x + \Delta y \, \partial U/\partial y + i \, \partial V/\partial x \, \Delta x + i \, \partial U/\partial y \, \Delta y}{\Delta x + i \Delta y}$$

THE DIFFERENTIAL COEFFIENTS MUST BE EXACTLY EQUAL WHEN $\Delta y = 0$ AND $\Delta x \to 0$ AND VICE VERSA. TWO EQUATIONS EVOLVE $f'(z) = \frac{\partial U}{\partial \cancel{y}} + i\frac{\partial V}{\partial x}$ AND $f'(z) = -i\left(\frac{\partial U}{\partial \cancel{x}} + i\frac{\partial V}{\partial y}\right)$

FOR THESE TO BE EQUAL WE MUST REQUIRE

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{AND} \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

THUS AN ANALYTIC FUNCTION MUST OBEY THESE EQUATION.

NOW TO THE DUMB PHYSICISTS WHO DOESN'T GIVE A DAMN ABOUT ALL THIS MATHEMATICS HE MAY COME ACROSS THESE ANALYTIC CRITERIAS IN A PROBLEM THAT REQUIRES SOLVING A DIFFERENTIAL EQUATION. HIS PROBLEM MAY HAVE THE PROPERTY THAT

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{AND} \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

THIS IS ACTUALLY A FAIRLY COMMON CONDITION FOR IF I DIFFERENTIATE AGAIN AND ELIMINATE V, I HAVE THAT U SATISFIES

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

THIS IS THE TWO DIMENSIONAL LAPLACIAN EQUATION. IN THREE DIMENSIONS IT WOULD BE

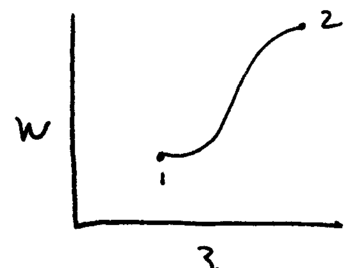
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

IF THE PROBLEM DOES NOT DEPEND ON z THEN THE SOLUTION IS NOT TOO DIFFICULT TO OBTAIN.

THE CAUCHY-RIEMAN CRITERIA HAS AN INTERESTING GEOMETRICAL SIGNIFICANCE. IF A MAPPING IS MADE ABOUT z_0 THE DERIVATIVE OF THE FUNCTION IS SHRUNK IN MAGNITUDED AND ROTATED THROUGH SOME ANGLE

COMPLEX INTEGRATION

Now I'd like to discuss integration in the complex plane. Suppose I want to go from 1 to 2. One way is to go along taking very small steps, i.e.



$$\int_1^2 f(z)\, dz = \sum f(z_i)\,\Delta z_i$$

We would write this as

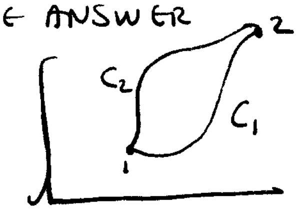
$$\sum \left[u(x_i,y_i) + i\,v(x_i,y_i)\right]\left[\Delta x_i + i\,\Delta y_i\right]$$

where in the integral we would take the real parts, i.e.,

$$\int_1^2 u\,dx - v\,dy$$

Now what determines the relative size of dx and dy? Only the curve. If the integration is made over a different path, then the answer would only depend on the end points, $F(z_2) - F(z_1)$. This is a consequence of F being analytic.

If we now have two separate paths between points 1 and 2 they should give the same answer



$$\int_{1\,C_1}^{2} = \int_{1\,C_2}^{2} \quad \text{OR} \quad \int_{C_1} + \int_{C_2} = 0$$

Thus the integral around a closed path is 0. $\oint u\,dx - v\,dy = 0$. This is a very important result to the physisists.

A more general result is

$$\oint a(x,y)\,dx + b(x,y)\,dy = \int_{\text{area}} \left(-\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x}\right) dx\,dy$$

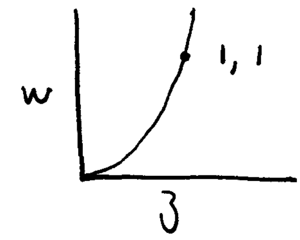
This is called Stokes' law. An example is

$$\oint F_x\,dx + F_y\,dy = \int_{\text{area}} \Big(\underbrace{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}}_{\text{curl } F}\Big)\,dx\,dy$$

LET'S TRY SOME PROBLEMS INVOLVING COMPLEX INTEGRATION.

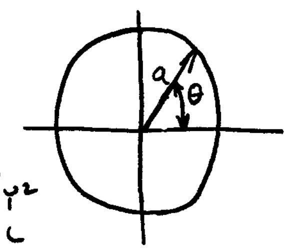
LET $w = z^2$ $u = x^2 - y^2$ AND $v = 2xy$

$$\int_0^1 u\,dx - v\,dy = \int (u-v)\,dx = \int_0^1 -2x^2\,dx = \frac{2}{3}$$



NOW LET'S TRY $w = 1/z$ AND INTEGRATE ABOUT A CIRCLE

$$\oint \frac{dz}{z} = \int \frac{x\,dx + y\,dy}{x^2 + y^2}$$



SINCE $w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$ $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

IF WE CHANGE VARIABLES THE PROBLEM WILL BE A LITTLE EASIER, THAT IS LET $z = ae^{i\theta}$.
a IS A CONSTANT RADIUS AND θ IS VARIABLE. WE HAVE THEN $dz = iae^{i\theta}d\theta$ AND

$$\int \frac{dz}{z} = \int \frac{iae^{i\theta}d\theta}{ae^{i\theta}} = \oint i\,d\theta = 2\pi i$$

NOTE THIS RESULT IS INDEPENT OF THE RADIUS. THIS RESULT IS VERY IMPORTANT. IT SAYS IF THE INTEGRAL CONTAINS A SINGULARITY THE VALUE OF THE INTEGRAL IS $2\pi i$. IF THERE ARE NO SINGULARITYES THEN THE CLOSED INTEGRAL IS 0.

$$\int \frac{dz}{z} = \ln z_1 - \ln z_2 = n2\pi i$$

WHERE n TELLS HOW MANY TIMES YOU GO AROUND.

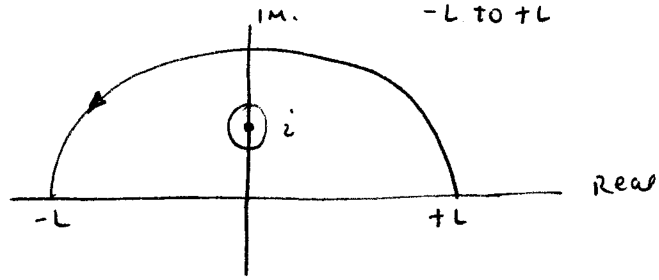
LET'S TRY $\oint \frac{dz}{z^2}$. LETTING $z = ae^{i\theta}$

$$\oint \frac{dz}{z^2} = \int \frac{ie^{-i\theta}}{a}d\theta = \frac{i}{a}\int \cos d\theta - \frac{i}{a}\int \sin\theta\,d\theta = 0$$

THUS $\int z^n dz = 0$ FOR n = INTEGER
$= 2\pi i$ " $n = -1$

LET'S DO ONE MORE PROBLEM

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{\substack{\text{real}\\ \text{LINE}\\ -L \text{ TO } +L}} \frac{dz}{1+z^2}$$



$$\Sigma = \int_{\text{LINE}} + \int_{\frac{1}{2}\odot} = 0$$

WE ONLY HAVE TO WORK OUT THE SEMI CIRCLE TO GET I. THUS

$$\int \frac{iLe^{i\theta}}{1+L^2 e^{2i\theta}}\, d\theta \simeq \frac{i}{L} \int e^{-i\theta}\, d\theta = 0 ?$$

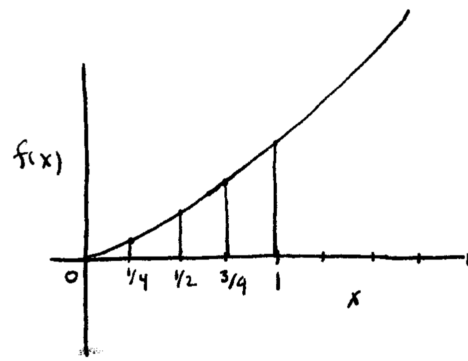
WHAT WENT WRONG? WELL, AT $z = i$ THE FUNCTION IS NOT ANALYTIC. THUS WE NEED THE INTEGRAL AROUND $z = i$

$$I = -\int_{\odot_i} \frac{dz'}{2iz'} = \frac{1}{2i}\, 2\pi i = \pi$$

NUMERICAL INTEGRATION
TRAPEZOIDAL AND SIMPSON RULE

Suppose we need the value of the integral of the function $\frac{1}{1+x^2}$ from 0 to 1. Of course we should all be able to directly integrate $\int_0^1 \frac{1}{1+x^2}dx = \tan x \Big|_0^1 = \tan 1 = \frac{\pi}{4}$. But suppose we weren't so small and we don't know how to integrate what do we do? Well, one way is to break up the interval into, say, 4 equal parts and evaluate the function $f(x) = \frac{1}{1+x^2}$ at each point, i.e.

x	f(x)
0	1.0
1/4	.941
1/2	.80
3/4	.64
1	.5



The trapezoidal rule tells us that the area under the curve is equal to

$$A = h\left[\tfrac{1}{2}f(0) + f(1/4) + f(1/2) + f(3/4) + 1/2\,f(1)\right]$$

where h = interval width

$$A = 1/4\left[1/2(1) + .94 + .8 + .64 + 1/2(1/2)\right]$$
$$= .7828$$

Since to exact answer is .78534 the trapezoidal rule is not so bad. It is in error by less than 1/2 %.

There is another method for evaluating the area under the integral and that involve the Simpson rule. The Simpson rule makes use of the general formula

$$A_s = \frac{h}{3}\left[1f(0), 4f(1), 2f(2), 4f(3), 2f(4), 4f(5), 1f(6)\right]$$

For the example above

$$A_s = \frac{1}{12}\left[1\times 1 + 4\times .941 + 2\times .80 + 4\times .64 + 1\times .5\right] = .78533$$

The result is amazingly accurate. It is in error by only 1 part in a hundred thousand. This is sensational accuracy for only 5 points spaced every .25 increment. Simpson's Rule is very powerful and should be used for most calculations which are difficult to integrate and yet which require accuracy.

As another example find

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = A$$

Let $h = 1/4$

x	$1+x^2$	$\ln(1+x)$	$\frac{\ln(1+x)}{1+x^2}$	
0	1.0	0	0	.000
1/4	.941	.18648	0.198	.792
1/2	.80	.40547	0.507	1.014
3/4	.64	.53357	0.832	3.228
1	.5	.69315	1.385	1.385
				6.419

$$A = \frac{1}{12}(6.419) = .535$$

In this example I have shown that instead of trying to integrate a hard function it is much easier to just work out some values by hand then use Simpson's rule to get the answer to high accuracy. Simpson's rule will work for almost all applications where accuracy less than 1 part in 10^6 is desired. It will not work when the integrand has a pole or when the integration limit produces infinity, i.e. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. One trick around this problem is to perform a perturbation analysis about the trouble point. In the case of $\int_0^1 \frac{\cosh x^2}{\sqrt{1-x^2}}\,dx$ we have trouble at $x = 1$. So let $x = 1-\epsilon$ and we have $\int_0^{.05} \frac{\cosh(1-\epsilon)^2}{\sqrt{1-(1-\epsilon)^2}}\,d\epsilon = \int \frac{\cosh(1-\epsilon)^2}{\sqrt{2\epsilon}\sqrt{1-\epsilon/2}}$

to first order we have $\frac{1}{\sqrt{2}}\int_0^{.05} \cosh 1 \frac{d\epsilon}{\sqrt{\epsilon}}$

For accuracy purposes the error in the Simbson rule result is given by $\mathcal{E} \approx \frac{2h^5}{180} f^{IV}(x)$

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MORE ON THE SIMPSON RULE

CONSIDER THE INTEGRAL

$$\int_0^\infty e^{-ax}\,dx$$

WHAT IS ITS VALUE? WELL, APPLYING SIMPSONS RULE IN STEPS OF 1 WE WOULD HAVE AN AREA WHICH DEPENDS ON THE VALUE OF a, I.E.

$$N(a) = \frac{1}{3}\left[\,1 + 4e^{-a} + 2e^{-2a} + 4e^{-3a} + \cdots\right]$$

WELL, THE INTEGRAL IS QUITE EASY TO EVALUATE AND IS JUST $1/a$. THUS WE HAVE THAT $N(a) = 1/a = 1/3\left[\,1+4e^{-a}+2e^{-2a}+4e^{-3a}+\cdots\right]$
THE SERIES CAN BE WRITTEN AS

$$\frac{1}{3}\left[\,1 + \frac{4e^{-a}}{1-e^{-2a}} + 2\,\frac{e^{-2a}}{1-e^{-2a}}\right]$$

I HAVE EVALUATED THE VALUE OF THE INTEGRAL FOR VARIOUS VALUES OF a USING INADDITION THE TRAPEZOIDAL RULE FOR COMPARISON.

a	$a\,N(a)$ TRAPEZOIDAL	$a\,(N(a)$ SIMPSON
ϵ	$1+\epsilon^2/12$	$1+\epsilon^4/180$
.5	1.021	1.00034
.69315	1.040	1.0012
1.00	1.082	1.0049
1.3863	1.155	1.0167
2.3025	1.407	1.093

NOTE FOR $a = .693$, 1.386 AND 2.30 THE FUNCTION IS DROPPING DOWN BY A FACTOR OF 2, 4, AND 10 RESPECTIVELY IN ONE e OR 2.72. IT IS INCREDIBLE THAT THE SIMPSON RULE PRESERVES HIGH ACCURACY EVEN TO $a = 1.3863$

DIFFERENTIAL OPERATOR, D

FOR REASONS which ARE NOT APPARENT I'd LIKE TO TALK About The DifferENTIAL OPERATOR, D, ANd SOME of ITS PROPERTIES. CONSIDER The FUNCTION $f(x)$ AND ITS deriVATIVE $f'(x)$, Then $f'(x) = \frac{d}{dx} f(x) = Df(x)$. D has MANY PROPERTIES which PERMIT ITS MANIPULATION AS AN ALGEBRAIC SYMBOL. For EXAMPLE $DDf(x) = f''(x) = D^2 f(x)$. ALSO

$$(1+2D+D^2)f(x) = f(x) + 2f'(x) + f''(x)$$

$$(1+D)^2 f(x)$$

Now LET'S ASK WHAT $\frac{1}{D}$ AND $\sqrt{D}$ MEAN. WE MAY GUESS THAT $1/D$ IS RELATED TO INTEGRATION AND THAT

$$D^{-1} = \frac{1}{D} = \int^x + C$$

The CONSTANT of INTEGRATION C IS NOT defined.

IN other words if D^{-1} OPERATES ON $f(x)$ Then IT produces $F(x)$. To PROVE This MULTIPLY Through bY D, i.e,

$$D\left(\frac{1}{D} f(x)\right) = DF(x) = f(x)$$

The SQUARE ROOT OPERATOR IS Defined AS

$$\sqrt{D}\left[\sqrt{D} f(x)\right] = f'(x)$$

The OPERATOR e^{aD} IS ONE of PARTICULAR IMPORTANCE AND IS Defined TO be

$$e^{aD} f(x) = f(x+a)$$

e^{aD} IS THUS CALLED A TRANSLATOR SINCE IT MOVES x to $x+a$. The EXPONENTIAL OPERATOR has The PROPERTY That $e^{bD}e^{aD} = e^{(a+b)D}$. To Prove This

$$e^{bD}e^{aD} f(x) = e^{(a+b)D} f(x)$$

$$e^{bD} f(x+a) = f(x+b+a)$$

$$f(x+a+b) = f(x+b+a)$$

LeT'S DIFFERENTIAL WITh RESPECT TO a

$D = \frac{d}{da}$

$$D e^{aD} f(x) = f'(x+a)$$

$$\frac{d}{dx}\left[e^{aD} f(x)\right]$$

$$\frac{d}{dx}\left(f(x+a)\right) = f'(x+a)$$

The operation $e^{aD} f$ has meaning when it is expanded

$$e^{aD} f(x) = \left(1 + aD + \frac{a^2 D^2}{2!} + \frac{a^3 D^3}{3!} + \cdots\right) f(x)$$

We can use the differential to derive the Simpson rule.
Let $I(x) = \int_0^x f(x)\,dx$. We want the integral at $x+2h$ to equal the following

$$I(x+2h) \approx I(x) + af(x) + bf(x+h) + cf(x+2h) + \text{error}$$

We need to find a, b, and c for this to be true. Thus we need to expand $I(x+2h)$ as a power series in h.

$$I(x+h) - I(x-h) = a f(x-h) + b f(x) + c f(x+h) + \xi$$

$$I(x) + hI'(x) + \frac{h^2}{2} I''(x) + \frac{h^3}{3!} I'''(x) - I(x-h) =$$

$$2h\,I'(x) + 2\frac{h^3}{6} I'''(x) + 2\frac{h^5}{5!} I^{v}(x) + \cdots = (a+b+c) f(x) + (ha-hc) f'(x)$$

Now $I'(x) \equiv f(x)$, $I'''(x) = f''(x)$, etc

f must match. Therefore

$$2h = a+b+c \qquad f$$

$$ha \pm hc \rightarrow a = c \qquad f'$$

$$\frac{h^3}{3} = h^2 a \rightarrow a = \frac{h}{3} \qquad f''$$

$$b = \frac{4h}{3}$$

Thus the pattern is $\frac{h}{3}(1,4,1)$

Using the differential operator notation we can work out the approximation

$$f(x + \tfrac{1}{2}h) \simeq a f(x-h) + b f(x) + c f(x+h) + d f(x+2h)$$

$$e^{\frac{1}{2}hD} = a e^{-hD} + b + c e^{+hD} + d e^{2hD}$$

Expanding as power series and matching terms

$$1 \simeq a e^{-\frac{3}{2}hD} + d e^{\frac{3}{2}hD} + b e^{-\frac{1}{2}hD} + c e^{+\frac{1}{2}hD}$$

$$a = d \quad \text{and} \quad b = c$$

$$1 = 2a\cosh(\tfrac{3}{2}hD) + 2b\cosh(\tfrac{1}{2}hD)$$

$$= 2a\left[1 + \frac{(\frac{3}{2}hD)^2}{2} + \frac{(\frac{3}{2}hD)^4}{24}\right] + 2b\left[1 + \frac{(\frac{1}{2}hD)^2}{2} + \frac{(\frac{1}{2}hD)^4}{24}\right]$$

We want $2a + 2b = 1 \qquad a = 1 - b$

$$2a\,\tfrac{9}{8} + 2b\,\tfrac{1}{8} = 0 \quad \rightarrow b = \tfrac{9}{16}, \quad a = -\tfrac{1}{16}$$

$$f(x) = -\tfrac{1}{16} f(x + \tfrac{3}{2}h) + \tfrac{9}{16} f(x + h/2) + \tfrac{9}{16} f(x - \tfrac{h}{2}) - \tfrac{1}{16} f(x - \tfrac{3h}{2})$$

MORE ON DIFFERENTIAL OPERATORS

We have talked about the differential operator D so now let me show you some of its interesting properties. If I have function $g(x)$ then the inverse operator, D^{-1}, working on it will produce the indefinite integral $\int^x g(u)du$, i.e,

$$D^{-1} g(x) = \int^x g(u)du$$

Extending this idea further the second integral of $g(x)$ is then

$$D^{-2} g(x) = \int^x (x-u) g(u) du$$

To prove this we can integrate by parts

$$D^{-2} g(x) = \underbrace{\int^x du}_{du} \; \underbrace{\int^u g(v) dv}_{V}$$

$$= uV|^x - \int^x u dV$$

$$= x\int^x g(v)dv - \int_c^x u g(u) du$$

collecting terms

$$D^{-2} g(x) = \int^x (g(u) (x-u) du$$

Therefore we find we can perform a double integration by a single integration. Now this idea can be generalized and we can show that

$$D^{-3} g(x) = \frac{1}{2}\int^x (x-u)^2 g(u) du$$

And in general

$$D^{-n} g(x) = \frac{1}{n!}\int^x (x-u)^{n-1} g(u) du$$

Thus using this formula we can give a meaning to the operation $D^{-1/2} g(x)$, i.e

$$D^{-1/2} g(x) = 1/\sqrt{\pi} \int^x (x-u)^{-1/2} g(u) du$$

The constant in front ~~is~~ may not be right but the result is most interesting.

SPECIAL FUNCTION ~~FUNCT~~ WORTH KNOWING

Suppose you had a problem which involved the function $F(x) = \int_0^x e^{-y^2}\,dy$. Well, you won't get far trying to evaluate this indefinite integral. If you can't integrate it; what do you do with it? Nothing! Well, not quite nothing. Others have tryed to integrate it and gone to a lot of trouble doing so. There are other functions similar to the above which are defined as integrals and as such have the property of carrying the function outside of the normal class of functions, viz, exponentials, logarithms, trigonometric, and transcendental functions. Only integration has this property of taking a function out of the normal class. Differentiation will not do this trick.

Some of these odd functions have come up frequently so people have gone to a lot of trouble to make up a set of tables which aids in evaluating the function. For example $F(x)$ is called the error function of x and is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,dy$$

This function frequently occurs in probability theory. It's old fashion to learn about these functions since anyone can look them up; however, it is worthwhile to know the behavior of the integral for various values of x.

So let's study $\operatorname{erf}(x)$ for large and small values of x. First for small x we can expand the exponential and integrate term by term

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - y^2 + \frac{y^4}{2!} - \frac{y^6}{3!} + \cdots\right) dy$$

$$= \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right]$$

This approximation will be valid for small x.

Now for large x the approximation is a little more difficult but we proceed as follows:

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-y^2}\,dy - \int_x^{\infty} e^{-y^2}\,dy\right.$$

The first term is definite and equals $\sqrt{\pi}/2$ Then we have

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-y^2}\,dy$$

Now we have to solve the integral. To do that we may write

$$\int_x^{\infty} e^{-y^2}\,dy = \int_x^{\infty} \frac{1}{2y}\underbrace{e^{-y^2}\,2y\,dy}_{d(-e^{-y^2})}$$

Integrating by parts

$$= \frac{1}{2x}e^{-x^2} - \int_x^{\infty} e^{-y^2}\,d\left(\frac{1}{2y}\right)$$

Now since x is large we can approximate x by y and expand the exponential and integrate term by term. We find that

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\frac{1}{2x}e^{-x^2}\left[1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \cdots\right.$$

Another way to arrive at this answer is to let $y = x+\eta$ where η is small then expand $e^{(x+\eta)^2}$, i.e.

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-(x+\eta)^2}\,d\eta = 1 - \frac{2}{\pi}\int_x^{\infty} e^{-x^2}e^{-2x\eta}e^{-\eta^2}\,d\eta$$

Dropping terms in η^2 for first approximations we can integrate $\int_x^{\infty} e^{-2x\eta}\,d\eta = \frac{1}{2x}$ so to first order

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\frac{1}{2x}e^{-x^2} + \text{higher order terms.}$$

The series which we have just written involves factorials in the numerator and therefore the series is ultimately diverging. Thus we may define the diverging series as having the value $\frac{\sqrt{\pi}}{2}\left[1 - \mathrm{erf}(x)\right] = 1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} + \cdots$

A very interesting property of the series for large x is that you can get away with a good approximation by retaining only the first few terms. ~~However~~, ultimately the series will diverge but you will have the right answer to a reasonable accuracy. The error in the result is of the order of the last term retains. For example if $x=2$ then the series is

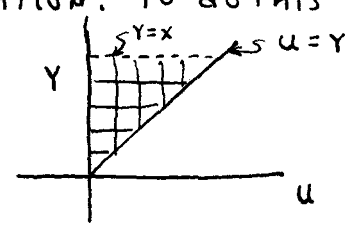
$$1-\frac{1}{8}+\frac{3}{64}-\frac{3\cdot 5}{8^3}+\frac{3\cdot 5\cdot 7}{8^4}+\cdots$$

if only 3 terms are kept, the answer is good to about 5%. If $x=1$ the error is quite large, of the order 1 itself so x must be greater that at least 2 for the approximation to work.

As a problem suppose you wanted the function given by the integral $\int_0^x \operatorname{erf}(y)\,dy$. This is not tabulated so you have to do some work. If we substitute for erf(y) we have the double integration

$$\int_0^x \int_0^y e^{-y^2}\,du\,dy$$

Now we want to reverse the order of integration. To do this it is worthwhile to draw a picture



We want the crossed hatch area integrating with respect to Y first.

$$\int_0^x\int_0^y e^{-u^2}\,du\,dy = \int_0^x\int_u^x e^{-u^2}\,dy\,du$$

$$= \int_0^x (x-u)\,e^{-u^2}\,du$$

$$= x\int_0^x e^{-u^2}\,du - \int u e^{-u^2}\,du$$

$$\int_0^x \operatorname{erf}(y)\,dy = x\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + \frac{1}{2}\left(e^{-x^2}-1\right)$$

ANOTHER PROBLEM WORTH SOLVING IS THE INTEGRAL

$$\int_{-\infty}^{\infty} e^{-x^2} e^{i\omega x}\, dx = \int_{-\infty}^{\infty} e^{-(x - i\omega/2)^2} e^{-\omega^2/4}\, dx$$

IF WE LET $y = x - \frac{i\omega}{2}$ THE INTEGRAL BECOMES

$$e^{-\omega^2/4} \int_{-\infty + i\omega/2}^{\infty + i\omega/2} e^{-y^2}\, dy$$

THE $i\omega/2$ ON THE INTEGRATION LIMITS CAN BE IGNORED SO WE HAVE THAT

$$\int_{-\infty}^{\infty} e^{-x^2} e^{i\omega x}\, dx = \sqrt{\pi}\, e^{-\omega^2/4}$$

WE FIND THAT INTEGRAL OF THE GAUSSIAN IS AN ERROR FUNCTION. AS AN EXERCISE PROVE

$$\int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{a^2}} \cdot e^{-\frac{x'^2}{b^2}} = \frac{\sqrt{\pi}}{\sqrt{a^2+b^2}}\, e^{-\frac{x^2}{a^2+b^2}}$$

A NUMBER OF PHYSICAL PHENOMENA INVOLVE INTEGRALS OF THE ABOVE TYPE. THE THEORY OF HEAT TRANSFER USES THE ABOVE INTEGRALS IN SOLVING THE DIFFERENTIAL EQUATION

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

THE SOLUTION TO THIS EQUATION IS

$$T = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x-x')^2}{4t}}$$

AN IMPORTANT PROPERTY OF THE DIFFERENTIAL EQUATION RESULTS IF $t \to 0^+$. IN THIS CASE T, THE TEMPERATURE, APPROACHES THE DELTA FUNCTION $\delta(x')$. THEREFORE IF AT $t=0$, T IS DESCRIBED BY $f(x)$ AT ANY LATER TIME t THE TEMPERATURE $T(x,t)$ IS GIVEN BY

$$T(x,t) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(x-x')^2}{4t}} f(x')\, dx'$$

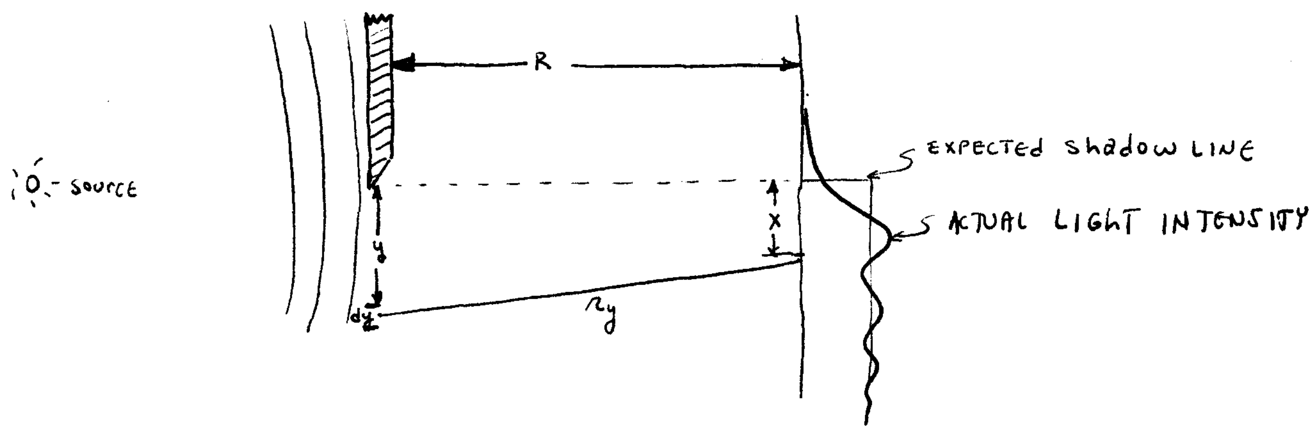
THIS IS A VERY IMPORTANT RESULT TO ALL OF MATH PHYSICS AND IT IS WORTH UNDERSTANDING ITS MEANING.

RELATED TO THE ERROR FUNCTION ARE SEVERAL OTHER SPECIAL FUNCTIONS ENCOUNTERED IN PHYSICS;

$$\int_0^x e^{y^2}\,dy \quad , \quad \int_0^x \cos y^2\,dy \quad , \int \sin y^2\,dy$$

THE LAST TWO ARE CALLED FRESNEL INTEGRALS AND ARE ENCOUNTERED IN OPTICS IN DIFFRACTION THEORY.

TO SEE HOW THE FRESNEL INTEGRALS EVOLVE AND ARE USEFUL CONSIDER A SINGLE SLIT OR KNIFE EDGE ILLUMINATED BY AN INFINITE SOURCE. THE SHADOW CAST BY THE SLIT IS NOT AS EXPECTED BY FALLS OFF IN AN OSCILLATORY FASHION WITH SOME LIGHT FALLING INSIDE THE SLIT



IN ORDER TO FIND THE LIGHT INTENSITY AT A DISTANT POINT X ON A SURFACE R FEET FROM THE KNIFE WE MUST USE HUYGEN'S PRINCIPLES WHICH REQUIRES US TO ADD ALL THE AMPLITUDES OF THE INCREMENTAL ELECTRIC FIELDS INCIDENT ON X. THUS WE HAVE THAT

$$A(t,x) = \int \cos\left[\omega\left(t - \frac{r_y}{c}\right)\right] dy$$

THE DISTANCE $r_y = \sqrt{R^2 + (y-x)^2}$ AND IS APPROXIMATELY $= R^2 + \frac{1}{2R}(x-y)^2$ SUCH THAT

$$A(t,x) = a \int \cos\left[\omega(t - R/c) + \frac{k}{2R}(x-y)^2\right] dy$$

WHERE $k = \omega/c = 2\pi/\lambda$ = WAVE NUMBER.

WRITING IN EXPONENTIAL FORM

$$A(t,x) = a\int e^{i\left[\omega(t-R/c) + \frac{k}{2R}(x-y)^2\right]}dy$$
$$= a\, e^{i\omega(t-R/c)}\int_0^{\infty} e^{\frac{ik}{2R}(x-y)^2}dy$$

THE INTEGRAL CAN BE WRITTEN IN THE FOLLOWING FORM

$$A(t,x) = a\sqrt{\frac{k}{2R}}\, e^{i\omega(t-R/c)}\left\{\underbrace{\int_{-x}^{x}\cos z^2\,dz}_{C} + i\underbrace{\int_{-x}^{\infty}\sin z^2\,dz}_{S}\right\}$$

where $z = (y-x)\sqrt{\frac{k}{2R}}$

THE LIGHT INTENSITY IS GIVEN BY THE SQUARE OF THE AMPLITUDE AND WE CAN WRITE

$$I = |A|^2 = \frac{a^2 k}{2R}\left[C^2 + S^2\right]$$

. WE CAN FIND I FOR LARGE VALUES OF x SINCE

$$C = \int_{-x}^{\infty}\cos y^2\,dy \simeq \int_{-\infty}^{\infty}\cos y^2\,dy = \frac{\sqrt{\pi}}{\sqrt{2}}$$

SIMILARLY $S = C = \sqrt{\pi}/\sqrt{2}$ THEN

$$I = \frac{I_0}{\pi}\left[C^2 + S^2\right]$$

where

$$C = \int_{-\sqrt{k/2R}\,x}^{\infty}\cos z^2\,dz = \int_{-\xi}^{\infty}\cos z^2\,dz$$

$$S = \int_{-\sqrt{k/2R}\,x}^{\infty}\sin z^2\,dz = \int_{-\xi}^{\infty}\sin z^2\,dz$$

where $\xi = x\sqrt{k/2R}$

IF WE WANT TO KNOW THE BEHAVIOR OF THE LIGHT INTENSITY INSIDE THE EDGE, I.E. FOR $\xi < 0$ WE WANT TO EVALUATE

$$\int_{+|\xi|}^{\infty}\cos z^2\,dz \quad \text{or} \quad \int_{|\xi|}^{\infty} e^{iz^2}\,dz = \int e^{iz^2}z\,dz\,\frac{1}{z}$$

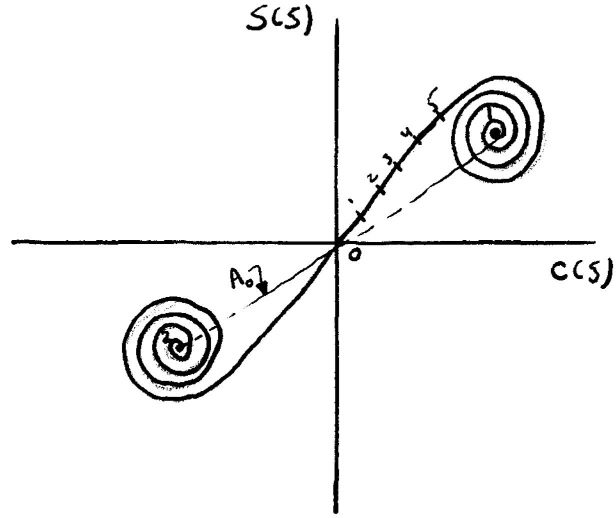
$$= \frac{1}{z}\left(-\frac{i}{2}e^{iz^2}\right)\Big|_{|\xi|}^{\infty} - \int_{|\xi|}^{\infty}\frac{i}{2}e^{iz^2}\frac{1}{z^2}\,dz =$$

If we look just at the first term we can get an idea of how the intensity falls off. Note that it is about equal to $\frac{i}{2|\xi|} e^{i|\xi|^2}$ to order $\frac{1}{\xi^2}$

Therefore
$$\int_{|\xi|}^{\infty} \cos z^2 \, dz \approx -\frac{\sin |\xi|^2}{2|\xi|}$$
$$\int_{|\xi|}^{\infty} \sin z^2 \, dz \approx \frac{\cos |\xi|^2}{2|\xi|}$$
Adding the squares to find the intensity we find it proportional to $1/4\pi\xi^2$. That is it falls off as $1/\xi^2$ going in from the edge.

On the other hand for large value of ξ from the edge we have that
$$\int_{-\xi}^{\infty} \cos z^2 \, dz = \int_{-\infty}^{\infty} \cos z^2 \, dz - \int_{-\infty}^{-\xi} \cos z^2 \, dz$$
$$= \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{\sin |\xi|^2}{2|\xi|}$$
$$\int_{-\xi}^{\infty} \sin z^2 \, dz = \sqrt{\pi/2} + \frac{\cos |\xi|^2}{2|\xi|}$$
Squaring and adding we find the intensity proportional to
$$\frac{\cos |\xi|^2 + \sin |\xi|^2}{\xi}$$
This describes an oscillatory intensity falling off as $1/\xi$. The oscillations are about some mean value proportional to the initial intensity.

There is an interesting property of the Fresnel integrals if we let
$$X = C(\xi) \quad \text{and} \quad Y = S(\xi)$$
and plot $C(\xi)$ vs $S(\xi)$ with ξ as a variable. We get a curve which looks like the following,



This curve is called a Cornu spiral. The curve has some interesting properties. Note that

$$dx = \cos \xi^2 \, d\xi \qquad dy = \sin \xi^2 \, d\xi$$

$$ds = \sqrt{dx^2 + dy^2} = d\xi$$

This means that ξ is a variable of length along the curve. The curve is layed out in equal units of ξ's. The slope of the curve is given by

$$\frac{dy}{dx} = \tan\theta = \tan \xi^2$$

or

$$\theta = \xi^2$$

Thus the curve has the very unique property that you keep turning at a rate equal to the square of ξ.

The distance between spiral centers is the amplitude of light thus the origin corresponds to the knife edge. The distance from 0 to 1 is $\frac{1}{2} A_0$ and the intensity is $\frac{1}{4} I_0$. As the length A changes or the spiral unwides it oscillates about the center.

MORE SPECIAL FUNCTIONS

EXPONENTIAL INTEGRALS

We have been discussing some of the special functions which you may come across in your work. Today I'd like to start with the function defined by the integral

$$\int^{x} \frac{e^{t}\,dt}{t} = E_i(x)$$

This indefinite integral of x is called the ~~exp~~ inverse exponential integral. For some stupid reason the exponential integral is defined as

$$\int_{x}^{\infty} \frac{e^{-t}\,dt}{t} = -E_i(-t)$$

The integral can be expressed as a sine and cosine function as

$$C_i(x) = -\int_{x}^{\infty} \frac{\cos t}{t}\,dt$$

$$S_i(x) = \int_{0}^{x} \frac{\sin t}{t}\,dt$$

Now there are tables of the integrals given for various values of x. You should not go to the trouble of trying to work them out. It is only important to recognize when you have reached this point in solving the problem so you can go to the tables.

While it is wasteful to try to integrate these functions, it is useful to investigate the behavior for small and large values of x. Therefore we will first look at what happens as $x \to 0$. We replace x by ϵ in the integral and ask what is the property of

$$I(\epsilon) = \int_{\epsilon}^{\infty} \frac{e^{-t}}{t}\,dt$$

To proceed we will add and subtract an integral which has the same behavior in thee limit as $\epsilon \to 0$.

An integral which can be used is $\int_\epsilon^1 \frac{1}{t}\,dt$ which is valid for small ϵ. This integral is definite an has the value $-\ln\epsilon$. Thus we can write

$$I = \int_\epsilon^\infty \frac{e^{-t}}{t}\,dt = \int_\epsilon^\infty \frac{e^{-t}}{t}\,dt - \int_\epsilon^1 \frac{1}{t}\,dt - \ln\epsilon$$

The integrals can be combined into the following form

$$I(\epsilon) = \int_1^\infty \frac{e^{-t}}{t}\,dt + \int_\epsilon^1 \frac{(e^{-t}-1)}{t}\,dt - \ln\epsilon$$

If we now replace ϵ by 0 the integrals are definite and have a finite value which is $-.5772$. This number is defined to be C, Euler's number. Thus we have then

$$I(\epsilon) = \int_\epsilon^\infty \frac{e^{-t}}{t}\,dt = -.5772 - \ln\epsilon = -\ln(1.781\epsilon)$$

The integral is logarithmic divergent and at a rate which depends on the value of ϵ.

The $Ci(x)$ and $Si(x)$ functions are necessarily logarithmically divergent for small x and have the limiting behavior for small x of

$$Ci(x) = \ln(1.781\,x)$$
$$Si(x) = x \qquad \text{for small } x$$

Euler's constant C is defined by the integral

$$C = \int_0^\infty e^{-x}\ln x\,dx = .5772$$

or in series notation

$$C = \lim_{n\to\infty}\left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} - \ln n\right]$$

There are several other properties of the exponential function worth knowing:

$$\int_0^\infty \frac{e^{-x}}{x+a}\,dx = e^{a}\left[-Ei(-a)\right]$$

$$\int_0^x Ei(u)\,dx = x\,Ei(-x) - (1-e^{-x})$$

As problems prove the following

$$\int_0^\infty e^{-pt}\, Si(qt)\, dt = \left(\frac{\pi}{2p} - \frac{1}{p}\tan^{-1}\frac{p}{q}\right)$$

And

$$\int_0^\infty \frac{\sin mt}{1+t^2}\, dt = \frac{1}{2}\left[e^{-m}\, Ei(m) - e^{+m}\, Ei(-m)\right]$$

The whole point of going into a discussion of these special functions is to be able to manipulate an integral like the lower one above into a form which can be evaluated by using tables. A good knowledge of these functions is quite valuable in working problems. As an example recall we had a problem where we were to find

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx$$

This is easy to evaluate if we let $\epsilon = 0$ and break it into two integrals, i.e.

$$I(a\epsilon) = \int_\epsilon^\infty \frac{e^{-ax}}{x} dx = \int_{a\epsilon}^\infty \frac{e^{-t}}{t} dt = -\ln(1.781\, a\epsilon)$$

where we let $x = t/a$. Likewise,

$$I(b\epsilon) = \int_\epsilon^\infty \frac{e^{-bx}}{x} dx = -\ln(1.781\, b\epsilon)$$

so we have

$$I(a\epsilon) - I(b\epsilon) = \ln(1.781\, b\epsilon) - \ln(1.781 a\epsilon) = \ln(b/a)$$

The more you learn the better your chances become of stumbling on the right way to get the answer. But remember one there is <u>A</u> way to find the answer there are immediately <u>MANY</u> ways to get the same answer.

Now lets study the behavior of $-Ei(-x)$ for large x. To start we can integrate $\int_x^\infty \frac{e^{-t}}{t} dt$ by parts,

$$-Ei(-x) = \left.\frac{e^{-t}}{t}\right|_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2} dt$$

Note here that we have just shown that

$$\int_x^\infty \frac{e^{-t}}{t^2} dt = Ei(-x) + \frac{e^{x}}{x}$$

Now we can integrate by parts again and get

$$-Ei(-x) = \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + \int_x^\infty \frac{2e^{-t}}{t^3}\,dt$$

We can keep this up and we will get a series which defines $-Ei(-x)$,

$$-Ei(-x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} + \cdots \right)$$

Observe that the coefficients are factorials and therefore for a given x you can go far enough that the series will ultimately diverge. If you ever ran across this series you should recognize it as $-e^{+x}\,Ei(-x)$

ELLIPTIC INTEGRALS

Another special function which is good to know is called the elliptic integral. Actually there are several kinds or classes of elliptic integrals.

The first kind of elliptic integrals is defined as,

$$F(k,\varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\varphi}} = u$$

This integral is not possible to integrate in terms of ordinary functions but rather involves transcendal functions.

The elliptic integral of the second kind is defined as,

$$E(k,\varphi) = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}\;d\theta$$

And the elliptic integral of the third kind is defined as

$$\int_0^\varphi \frac{d\theta}{(1+n^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

The first two kinds of elliptic integrals are more frequently encountered and therefore I will discuss them.

One special case of the 1st kind elliptic integral is for the case when φ varies between 0 and $\pi/2$. This function is called $K(k)$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

This is called the complete elliptic integral.

Other forms of the elliptic integrals may require substitution of variables in order to recognize. One common substitution is $t = \sin\theta$. The differential substitution is $d\theta = \frac{dt}{\sqrt{1-t^2}}$ so that $u = F(k,\varphi)$ becomes

$$u = \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

Further generality can be obtained by shifting the scale of t such that we have

$$u = \int \frac{dt}{\sqrt{a^2-b^2t^2}\,\sqrt{c^2-d^2t^2}}$$

or in polynomial form

$$u = \int \frac{dt}{\sqrt{a+bt+ct^2+dt^3+et^4}}$$

If the variable of integration t is greater than a quadratic, you are out of luck trying to cast the integrate in elliptic form. However you should try to complete the square and see if the polynomial will factor.

Sometimes the inverse elliptic functions are used. In this notation $\varphi = f(u)$ and is called $\operatorname{am} u$. The transformation equations are

$$\cos\varphi = \operatorname{cn}(u)$$
$$\sin\varphi = \operatorname{sn}(u)$$
$$\sqrt{1-k^2\sin^2 u} = \operatorname{dn}(u)$$

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You may be wondering what elliptic functions are in the mathematical world. You know that trignometric functions such as the sine and cosine are the simplest periodic functions from which all other common functions can be expressed as in series notations, e.g exponentials. When operating in the complex plane the functions sn(u) and cn(u) have the property of being the simplest doubly periodic functions which are analytic. They therefore form the base for transcendal functions which can be expressed in terms of sn(u) and cn(u). These are very interesting functions to the mathematician.

It is worth discussing problems which involve elliptic integrals so you get a feel for the appearance in physics and engineering. Originally they were used to determine the length of an ellipse, i.e, its perimeter.

In the analysis of a swinging pendulum the governing differential equation is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

The period of oscillation is dependent on the amplitude of oscillation. Only when $\sin\theta \approx \theta$ is the period independent of amplitude. The above differential equation is solved through the use of elliptic integrals.

Now because all this is very boring to learn about these special functions I'll give you an interesting problem where these weird functions come up. This problem was found, by the way, after the answer was known.

Suppose I want the mean of two numbers say 1 and 9. Well, the answer is easy; the arimethmic mean is 5. But, wait! What about the geometric means? It is 3. Which is the better? Suppose I want the mean mean or the great mean. Why not take the mean of 3 and 5 which is 4? And then the geometric mean is 3.873.

Now we can keep going until we get what is called the Gauss's mean. This may be called the Arithmetic geometric mean of two number m, n. It turns out that this mean is given by

$$\text{mean}(m,n) = \frac{\pi/2}{\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2\sin^2\theta + n^2\cos^2\theta}}} = \frac{\pi m}{2K\left(\sqrt{1-\frac{m^2}{n^2}}\right)}$$

You would be hard pressed to associate such a difficult function with such a simple idea as computing a mean but it is true.

BESSEL FUNCTIONS

To amuse you further I'll introduce you to a new set of special functions in a useless but interesting way. Some of you have heard of continuing fractions. They are of the form

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} = X$$

This fraction is of the form

$$X = 2 + \frac{1}{X}$$

which is just a quadratic equation in X. Now we can vary the ~~Rhymth~~ Rythm of the fraction by changing the repetitive numbers, e.g

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1+2}}}} = X$$

Now the form is $X = 1 + \frac{1}{2+\frac{1}{X}} = 1 + \frac{X}{2+X}$

which is again a quadratic in X. No matter what repetition pattern you pick you will always end up with a quadratic relationship.

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Now it is possible to construct a continuing fraction which does not have the rhythmic repetition of the above fraction. The simplest non rhythmic fraction is the following

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5 + 1}}}}$$

Finding the formula for this fraction is quite difficult. In fact it held up the development of the atomic bomb while we worked it out. With considerable effort you can show the above fraction is equal to

$$\frac{-2i\, J_1(-2i)}{J_0(-2i)}$$

Here the symbols J_1 and J_0 refer to Bessel functions so we see that these special functions can be related to ordinary numbers.

Bessel functions are defined as definite integrals in the following manner

$$J_n(v) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - v\sin t)\, dt$$

n here is an integer index with v being a variable. There are many ways to express the Bessel function; one more common way is

$$J_n(v) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iv\cos t}\, e^{int}\, dt$$

Bessel functions appear many times when the integral involves the product of sines and cosines. A lot of physics problems turn up Bessel functions; one being in the analysis of frequency modulation. In this case we have a transmitted wave given by $A\cos\varphi(t)$. The phase $\varphi(t)$ is time varying as the carrier frequency is modulated by the signal frequency, ν, i.e, the wave frequency, ω is $\omega_0 + a\sin(\nu t)$.

SINCE THE PHASE IS CHANGING IT DEPENDS ON THE INSTANTEOUS FREQUENCY. WE NEED TO SOLVE $\frac{d\varphi}{dt} = \omega(t)$ OR

$$\varphi = \omega_0 t + \frac{a}{\nu} \cos \nu t$$

THUS THE SIGNAL IS DESCRIBED AS

$$A \cos\left(\omega_0 t + \frac{a}{\nu} \cos \nu t\right)$$

NOW WE MIGHT ASK WHAT FREQUENCY COMPONENTS ARE THERE IN THIS WAVE, I.E. WHAT SIDE BAND FREQUENCIES ARE CARRIED? WE NEED TO EVALUATE THE INTEGRAL

$$\int \cos\left(\omega_0 t + \frac{a}{\nu} \cos \nu t\right) e^{i\nu t}\, dt$$

TO DETERMINE HOW MUCH OF FREQUENCY ν IS CARRIED. THIS THEN IS AN EXAMPLE OF THE USE OF A BESSEL FUNCTION.

WHENEVER THE PROBLEM INVOLVES TWO DIMENSIONAL WAVE PROPAGATION OR MOTION BESSEL FUNCTIONS APPEAR. SUCH PROBLEMS AS WATER WAVES, DRUM HEAD VIBRATION AND OTHER PROBLEMS INVOLVING CYLINDRICAL SYMMETRY INVOLVE SOLVING BESSEL FUNCTIONS.

PROPERTIES OF BESSEL FUNCTIONS

I'D NOW LIKE TO DISCUSS SOME OF THE PROPERTIES OF BESSEL FUNCTIONS. THE SIMPLEST FORM OF THE BESSEL FUNCTION $J_N(V)$ IS WHEN THE INDEX $n = 0$ AND WE HAVE THE ZERO ORDER BESSEL FUNCTION,

$$J_0(V) = \frac{1}{2\pi} \int_0^{2\pi} e^{iV\cos t}\, dt$$

THE n^{th}, $n+1$ AND $n-1$ ORDER BESSEL FUNCTIONS CAN BE RELATED THROUGH A RECURSION RELATIONSHIP OF THE FORM

$$n J_n(V) = \frac{V}{2}\left[J_{n+1}(V) + J_{n-1}(V)\right]$$

OR EXPRESSED ANOTHER WAY

$$J_{n+1}(V) = \frac{2n}{V} J_n(V) - J_{n-1}(V)$$

THEREFORE IF WE HAVE A TABLE OF J_0 AND J_1 THEN WE HAVE ALL HIGHER ORDER BESSEL FUNCTIONS BY USING THE ABOVE FORMULA. THIS SAVES A LOT OF WORK.

SOME OTHER USEFUL RELATIONSHIP OF BESSEL FUNCTIONS

The derivative of the Bessel function is given by

$$\frac{d}{dv} J_n(v) = J_n'(v)$$

by differentiating under the integral sign we have

$$J_n'(v) = \frac{1}{\pi}\int_0^{\pi} (-\sin t)\sin(nt - v\sin t)\, dt$$

And by expanding the product of the sines as the difference of 2 cosines we can write

$$J_n'(v) = -\frac{1}{2}\left[J_{n+1}(v) - J_{n-1}(v)\right]$$

Other properties of the derivative of the Bessel function are:

$$\frac{d}{dv}\left[v^n J_n(v)\right] = v^n J_{n-1}(v)$$

$$\frac{d}{dv}\left[v^{-n} J_n(v)\right] = -v^{-n} J_{n+1}(v)$$

$$\int J_1(v)\,dv = -J_0(v)$$

$$\int v\,J_0(v)\,dv = v\,J_1(v)$$

$$\int \left[J_0(v)\right]^2 v\,dv = \frac{v^2}{2}\left[J_0^2(v) + J_1^2(v)\right]$$

MORE ON BESSEL FUNCTIONS

LAST TIME I WAS DISCUSSING BESSEL FUNCTIONS AND HAD ENUMERATED SOME OF THE MORE USEFUL RELATIONSHIPS WORTH REMEMBERING. THEY WERE:

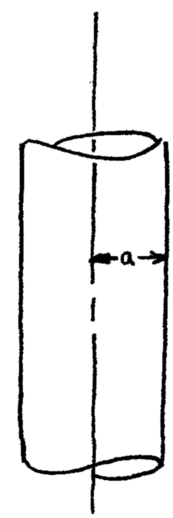
$$J_n(v) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{int} e^{iv\cos t}\, dt$$

$$n J_n(v) = \frac{v}{2} \left(J_{n+1} + J_{n-1} \right)$$

$$J_n'(v) = J_{n-1} - \frac{n}{v} J_n$$

$$J_n'' + \frac{1}{v} J_n' + \left(1 - \frac{n^2}{v^2}\right) J_n = 0$$

BESSEL FUNCTIONS APPEAR FREQUENTLY IN PHYSICS AND ONE OF THE MOST COMMON OCCURENCES IS IN THE FORM OF THE SOLUTION TO A DIFFERENTIAL EQUATION. SINCE THE DIFFERENTIAL EQUATION IS IMPORTANT IN PHYSICS, I'LL ATTEMPT TO WORK A PROBLEM.



SUPPOSE THAT WAVES ARE PROPAGATING INSIDE A CYLINDER OF RADIUS a. THE WAVES SATISFY THE WAVE EQUATION GIVEN BY

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

WHERE ∇^2 IS THE LAPLACIAN OPERATOR,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

LET ME FIRST CONSIDER THE CASE OF AN INFINITE CYLINDER AND THE WAVES ARE RUNNING UP AND DOWN. I WANT A SOLUTION WHICH IS OF THE FORM

$$\psi(x,y,z,t) = e^{i\omega t} F(x,y,z)$$

IN THIS SOLUTION I AM LOOKING FOR A SOLUTION WITH A DEFINITE FREQUENCY, ω. I ALSO WANT A PARTICULAR z TRAVELING WAVE WHICH HAS THE PROPERTY AT THE WALL IT GOES TO ZERO, I.E, $\psi = 0$ AT $\rho = a$. FOR CONVENIENCE IT IS EASIER TO WORK IN POLAR COORDINATES DUE TO THE SYMMETRY OF THE PROBLEM.

The solution would be of the form

$$\psi(x,y,z,t) = e^{i\omega t} e^{iKz} G(\rho\cos\varphi, \rho\sin\varphi)$$

where I transformed x as $\rho\cos\varphi$ and y as $\rho\sin\varphi$. Now I can differentiate and get the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G = \left(\frac{\omega^2}{c^2} - K^2\right) G \equiv \lambda^2 G$$

λ is sometimes ~~to~~ referred to as the eigenvalves of the differential equation. Note if I had no z variation, then $\lambda = \omega/c$. Now to go further I must transform the differential operator to polar coordinates or variables ρ and φ. This transformation is accomplished using the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

Working ~~the~~ through the algebra yields the result

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

Notice there are no sines or cosines in the equation. This is as it should be, since the problem is symmetric about the z axis. Now we need to solve the equation

$$\frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \varphi^2} = \lambda^2 G$$

Assuming now form the moment that $\lambda = \omega/c$ and I have no z variation I shall try a solution of the form

$$G(\rho, \varphi) = F(\rho)\, \Phi(\varphi)$$

There is really no good reason why I try this solution other than I know it works. However, it is only when special geometries permit that separation of variables is worth trying. With I plug this solution into the differential equation I have

$$\Phi\left(F'' + \frac{1}{\rho}F'\right) + \frac{F}{\rho^2}\Phi'' = -\frac{\omega^2}{c^2} F\Phi$$

Here's where people argue a little because we will divide through by $F\Phi$. I guess it's okay to do that because it works!

IF I DIVIDE THROUGH BY $F\Phi$ I GET

$$\frac{F'' + (1/\rho)F'}{F} + \frac{\Phi''}{\rho^2\Phi} = -\frac{\omega^2}{c^2}$$

NOW HERE IS AN INTERESTING POINT. THE TWO TERMS ON THE LEFT HAVE UNIQUE PROPERTIES. THE TERM IN F IS ONLY A FUNCTION OF ρ WHILE THE TERM IN Φ IS ONLY A FUNCTION OF φ. IN PARTICULAR Φ MUST BE A VERY SPECIAL FUNCTION WHICH SATISFIES THE EQUATION

$$\frac{\Phi''}{\rho^2\Phi} = -n^2$$

WHERE n IS SOME CONSTANT. THE SOLUTION TO THIS EQUATION IS JUST

$$\Phi(\varphi) = e^{in\varphi} \quad \text{OR} \quad \Phi(\varphi) = e^{-in\varphi}$$

THE MOST GENERAL SOLUTION WOULD BE

$$\Phi(\varphi) = A\sin n\varphi + B\cos n\phi$$

THE SOLUTION MUST BE PERIODIC THEREFORE n CANNOT BE LESS THAN 0 OR IMAGINARY. THE FUNCTION MUST SATISFY THE PHYSICAL CONSTRAINT THAT ~~THAT~~ WHEN YOU GO AROUND 360° YOU MUST HAVE THE SAME FUNCTION E.G., YOU MUST READ THE SAME PRESSURE OR DISPLACEMENT.

KNOWING THE SOLUTION FOR Φ, THE EQUATION FOR F CAN BE SOLVED,

$$F''(\rho) + \frac{1}{\rho}F'(\rho) - \frac{n^2}{\rho^2}F(\rho) = -\frac{\omega^2}{c^2}F(\rho)$$

IT IS CONVENIENT TO CHANGE THE SCALE BY LETTING $\rho = \frac{c}{\omega}r$ SUCH THAT $F(\rho) \rightarrow J(\frac{\omega}{c}r)$. THEN WE CAN WRITE

$$\frac{d^2J(r)}{dr^2} + \frac{1}{r}\frac{dJ(r)}{dr} + \left(1 - \frac{n^2}{r^2}\right)J(r) = 0$$

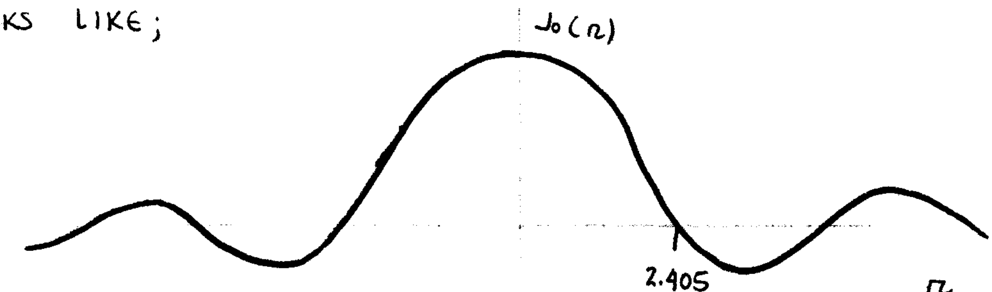
$J_n(r)$ IS NOW THE BESSEL FUNCTION OF THE VARIABLE r AND CONSTITUTES A SOLUTION TO THE DIFFERENTIAL EQUATION. THUS WE HAVE AS A COMPLETE SOLUTION

$$G = e^{in\varphi}e^{i\omega t}J_n(r)e^{ik_z z}$$

IN THIS SOLUTION THE BESSEL FUNCTION TAKES THE PLACE OF THE SIMPLE SINE, COSINE VARIATION IN A ONE DIMENSIONAL ~~PLA~~ PROBLEM. THE BESSEL FUNCTION IS THE TWO DIMENSIONAL ANALOG OF THE SINES AND COSINES.

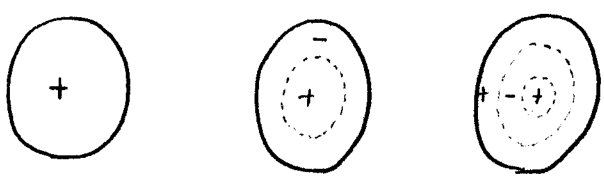
THE BESSEL FUNCTIONS HAVE SOME INTERESTING PROPERTIES AND WE SHOULD TAKE A LOOK AT SOME OF THEM.

THE SIMPLEST WAVE OR MODE OF THE BESSEL FUNCTION DOES NOT DEPEND ON φ SO THAT $n = 0$. THE FUNCTION $J_0(r)$ LOOKS LIKE;



$r = \mathcal{K}\rho = \rho\left(\omega^2/c^2 - K_z^2\right)^{\frac{1}{2}}$. IF WE REQUIRE $\Phi = 0$ AT $\rho = 0$ THEN $J_0(\mathcal{K}a) = 0$ IMPLIES THAT $\mathcal{K} = 2.405/a$. NOW IF ω/c IS LESS THAN $2.405/a$ THE EXPONENTIAL IS NEGATIVE AND REAL AND THE WAVE WILL NOT PROPAGATE INTO THE PIPE. ON THE OTHERHAND IF THE FREQUENCY IS HIGH ENOUGH, THE WAVE GETS TRANSMITTED.

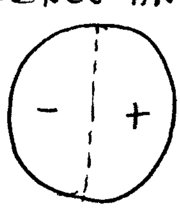
IN THE CASE OF A DRUM HEAD $K_z = 0$ AND THE FIRST NATURAL FREQUENCY IS $\omega = \frac{2.405}{a}c$. HIGHER FREQUENCIES CORRESPOND TO HIGHER ORDER ZEROES OF THE BESSEL FUNCTION. THE ROOTS ARE NOT IN ANY SIMPLE MULTIPLE RELATIONSHIP THEREFORE THE NOTES FROM A DRUM ARE NOT MELODIUS. A PICTURE OF THE FIRST 3 MODES OF THE ZERO ORDER BESSEL FUNCTION ARE:

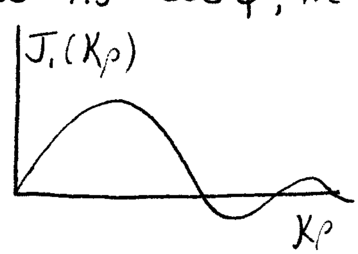


THE + DENOTES AN UPWARD MOTION; − DENOTES A DOWNWARD MOTION. THE DASH LINES DENOTE NODES

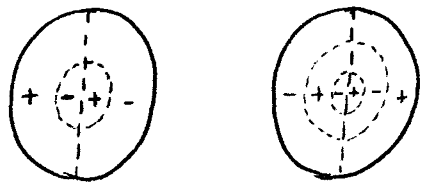
THE LOWEST MODE OF THE FIRST ORDER BESSEL FUNCTION INVOLVES A φ DEPENDENCE AND VARIES AS $\cos\varphi$, I.E

$J_1(\mathcal{K}\rho)$





THE NEXT TWO MODES OF J_1 LOOK LIKE



WE WANT TO INVESTIGATE SOME MORE PROPERTIES OF THE BESSEL FUNCTION, $J_n(v)$

$$J_n(v) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{int} e^{iv\cos t} dt$$

LET'S CONSIDER THE BEHAVIOR OF $J_n(v)$ FOR SMALL V. TO PROCEED WE'LL EXPAND $e^{iv\cos t}$ AS A POWER SERIES. ALSO I'LL JUST DO $J_0(v)$ FOR NOW.

$$J_0(v) = \frac{1}{2\pi} \int_0^{2\pi} e^{iv\cos t} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_k \frac{i^k v^k \cos^k t}{k!} dt$$

$$= \frac{1}{2\pi} \sum \frac{i^k v^k}{k!} \int_0^{2\pi} \cos^k t \, dt$$

NOW I DON'T KNOW HOW TO INTEGRATE THE FUNCTION $\cos^k t$. FIRST I'LL SEE IF I UNDERSTAND IT TERM BY TERM

k	$\cos^k t$	$\int_0^{2\pi} \cos^k t \, dt$
0	1	2π
1	$\cos t$	0
2	$\cos^2 t$	π
3	$\cos^3 t$	0
4	$\cos^4 t$	$\frac{1}{4}\pi$

WELL IT LOOKS LIKE I MAY BE ABLE TO SOLVE THIS. I ALREADY KNOW k MUST BE EVEN OTHERWISE THE INTEGRAL EQUAL ZERO. THEREFORE LET $k = 2\ell$; ℓ BEING AN INTEGER. NOW I WANT TO INTEGRATE $\int_0^{2\pi} \cos^{2\ell} t \, dt$. REPLACE THE COSINE BY EXPONENTIALS,

$$\frac{1}{2^{2\ell}} \int_0^{2\pi} [e^{it} + e^{-it}]^{2\ell} dt.$$

NOW WE WANT TO WRITE THE INTEGRAND AS

$$\frac{1}{2^{2l}}\left[e^{2ilt} + e^{-2i(l-1)t}\,e^{+2it}\right]$$

AND USE THE BINOMIAL THEOREM WHICH SAYS

$$\Sigma\,(x+y)^n = \sum_k \frac{n!\,x^{n-k}\,y^k}{(n-k)!\,k!}$$

WHERE $x = e^{2ilt}$ AND y

WE NOW NEED $\int_0^{2\pi} e^{nit}\,dt = \frac{1}{ni}\,e^{nit}\Big|_0^{2\pi} \begin{matrix} = 0 & \text{if } n \neq 0 \\ = 2\pi & \text{if } n = 0 \end{matrix}$

RETAINING THE CENTRAL TERM IN THE EXPANSION

$$\int = \frac{2l!}{2^{2l}}\,\frac{2\pi}{l!\,l!}$$

THUS

$$J_0(v) = \sum_l \frac{(-1)^l\,v^{2l}}{\cancel{(2\pi)}\,\cancel{(2l)}!}\;\frac{\cancel{(2l)}\;\cancel{2\pi}}{2^{2l}\,l!\,l!} = \Sigma\,\frac{(-1)^l}{l!\,l!}\left(\frac{v}{2}\right)^{2l}$$

$$= 1 - \frac{(v/2)^2}{1\cdot 1} + \frac{\left(\frac{v}{2}\right)^4}{2!\,2!} - \frac{\left(\frac{v}{2}\right)^6}{3!\,3!}$$

THIS SERIES BEHAVES AS A COSINE FUNCTION.

GENERALIZING THIS RESULT TO THE N^{th} ORDER BESSEL FUNCTION IS

$$J_n(v) = \sum_n \frac{v^n}{2^n n!}\left[1 - \frac{(v/2)^2}{n+1} + \frac{(v/2)^4}{2!\,(n+1)(n+2)} - \frac{(v/2)^6}{3!\,(n+1)(n+2)(n+3)} + \cdots\right.$$

SO FAR WE HAVE ONLY DEALT WITH INTEGER VALUES OF n BUT LET'S NOW ASK WHAT IF $n = 1/2$. THEN WHAT IS $J_{1/2}(v)$? WELL IMMEDIATELY WE HAVE TROUBLE BECAUSE WE DON'T KNOW WHAT $(1/2)!$ IS. BUT DAMN THE TORPEDOES THAT IS ONLY A SCALE FACTOR; LET'S GO AHEAD AND LET $n = 1/2$ AND SEE WHAT WE HAVE,

$$J_{1/2}(v) = \frac{v^{1/2}}{\sqrt{2}\,(1/2)!}\left[1 - \frac{v^2}{2^2(3/2)} + \frac{v^4}{2^4 2!(3/2)(5/2)} - \frac{v^6}{2^6 3!(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}\right.$$

THE DEMONINATORS CAN BE REARRANGED SUCH THAT WE HAVE

$$J_{1/2}(v) = \frac{v^{1/2}}{\sqrt{2}\,(1/2)!}\left[1 - \frac{v^2}{3!} + \frac{v^4}{5!} - \frac{v^6}{7!} + \cdots\right]$$

The series is just the $\sin v$ so we have within a constant

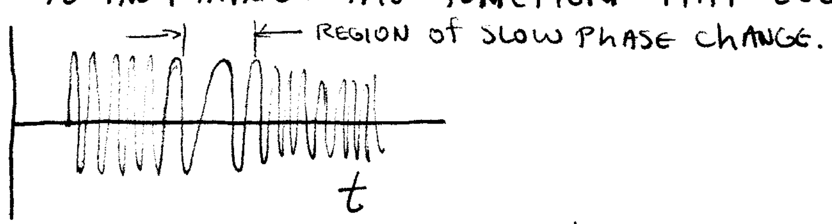
$$J_{1/2}(v) = \frac{\sin v}{\sqrt{v}}$$

The constant turns out to be $\sqrt{2/\pi}$. Isn't it amazing that $J_{1/2}(v)$ turns out to be such a simple ordinary function after all this higher mathematical hokus pokus. Note further that since we have a recursion formula that we can compute $J_{3/2}(v)$ and other half order Bessel functions.

Method of Steepest Descent

I next want to talk about the behavior of the Bessel function for large v. This is an interesting case ~~and~~ because we again have to expand as a power series but now be careful in which terms we keep. We will use a method of integration which is very useful called the method of steepest descent.

The idea is involved with an integral like $\int_0^{2\pi} e^{int} e^{iv\cos t}\,dt$ where v may be 10,000 or more. Now the function oscillates like hell and never gets anywhere. It is only when integrating through a small range of t where the phase is not changing so rapidly that the integral has a chance to amount to anything. The function may look like



We have then the integral $\int_0^{2\pi} e^{i\phi(t)}\,dt$ where $\phi(t) = nt + v\cos t$ and we want to know where ϕ is a max or min. It is only at the extremum that the integral can amount to anything. Thus we want $\phi'(t_0) = 0$ where t_0 is the time the phase is not changing.

DIFFERENTIATING $\phi(t)$ AND SOLVING for t_0

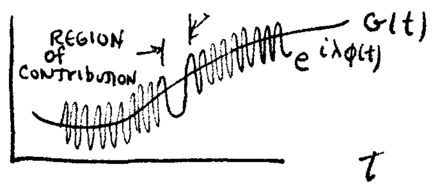
$$V \sin t_0 + n = 0$$

$$t_0 = \sin^{-1} 2\pi - \frac{n}{V}$$

LET ME GO OVER THIS AGAIN IN A MORE GENERAL MANNER SINCE IT IS IMPORTANT. SUPPOSE I WANT

$$I = \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} G(t)\, e^{i\lambda\phi(t)}\, dt$$

WHERE $G(t)$ IS ANY SMOOTH FUNCTION of t.



NOW WE WANT $\phi(t)$ EXPANDED AS A POWER SERIES ABOUT t_0

$$\phi(t) = \phi(t_0) + (t-t_0)\,\phi'(t_0) + \frac{(t-t_0)^2}{2}\,\phi''(t_0)$$

WHERE t_0 MAKE $\phi'(t_0)$ 0. SUBSTITUTING BACK INTO I

$$I = \int G(t_0)\, e^{i\lambda\phi(t_0)}\, e^{i\lambda\phi''(t_0)\left(\frac{t-t_0}{2}\right)^2}\, dt$$

THE EXPONENTIAL $e^{i\lambda\phi''(t_0)\left(\frac{t-t_0}{2}\right)^2}$ IS A GAUSSIAN OSCILLATOR AND HAS THE CHARACTER

$$\int e^{-\alpha t^2}\, dt = \sqrt{\frac{\pi}{\alpha}}$$

HERE $\alpha = -i\lambda\phi''$. THEN AS AN APPROXIMATION

$$\int_{-\infty}^{\infty} G(t)\, e^{i\lambda\phi(t)}\, dt \approx \frac{G(t_0)\, e^{i\lambda\phi(t_0)}}{\sqrt{(-i)\lambda\,\phi''(t_0)}}$$

IN OUR CASE $G = e^{int}$, $\lambda = V$ AND $\phi = \cos t$ WITH $t_0 = 0$ or 2π. $G(t_0) = 1$, $\phi''(t_0) = -1$

$$\int_0^{2\pi} e^{int}\, e^{iV\cos t}\, dt = \frac{\sqrt{i}}{\sqrt{V}}\, e^{iV} = \frac{\sqrt{i}}{\sqrt{V}} \cos V - \frac{\sin V}{\sqrt{V}}$$

If t HAS SEVERAL MAXIMA, THEN YOU JUST SUM UP ALL THE CONTRIBUTIONS.

MORE ON THE METHOD OF STATIONARY PHASE

LAST TIME WE WERE TRYING TO FIND THE BEHAVIOR OF THE INTEGRAL $\int f(t) e^{i\lambda F(t)} dt$ FOR LARGE VALUES OF λ. IN PARTICULAR THE INTEGRAL WAS THE BESSEL FUNCTION,

$$J_n(v) = \frac{1}{2\pi i^n} \oint_{-\pi \text{ to } \pi} e^{iv\cos t} e^{int} dt$$

THE VARIABLE v IS THE LARGE VARIABLE λ AND WE ARE INTERESTED IN WHAT THE VALUE OF THE INTEGRAL IS. THE ONLY TIME THE INTEGRAL AMOUNTS TO ANYTHING IS WHEN THE PHASE IS CONSTANT OR SAID MATHEMATICALLY WHEN

$$\frac{d}{dt}(v\cos t) = 0$$

THIS CONDITION IMPLIES THAT $v\sin t = 0$ OR $t = 0$ OR π.

SINCE THE INTEGRAL ONLY CONTRIBUTES NEAR $t=0$ WE CAN EXPAND $\cos t$ AS $1 - t^2/2$ THUS

$$\int_{\epsilon-6^\circ}^{2\pi-6^\circ} e^{iv} e^{-ivt^2/2} e^{int} dt$$

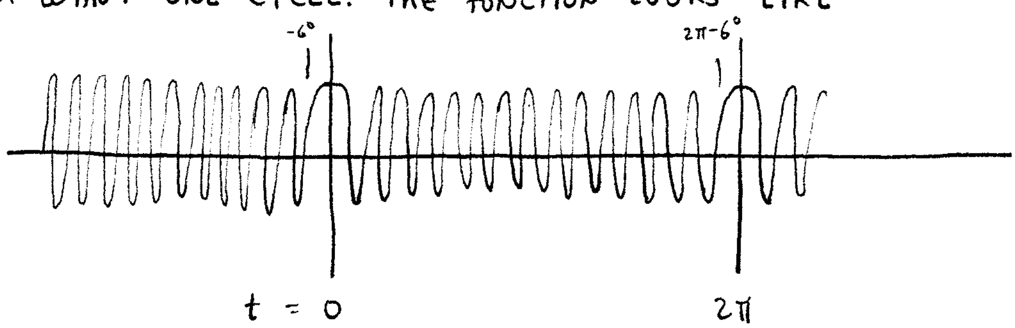
HERE THE INTEGRATION LIMITS HAVE BEEN ADJUSTED SO WE DON'T HAVE TO WORRY ABOUT THE FUNCTION RIGHT AT $t=0$. IF WE NOW APPROXIMATE $e^{int} \approx e^{in0} = 1$ AND RECALL

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \sqrt{\pi/\alpha}$$

WE THEN HAVE

$$e^{iv} \int_{-6}^{2\pi-6^\circ} e^{-ivt^2/2} dt = e^{iv}\sqrt{\frac{2\pi}{iv}} = e^{iv} e^{-i\pi/4}\sqrt{\frac{2\pi}{v}}$$

NOW WE HAVE TO BE CAREFULL BE THE INTEGRAL FROM $-\infty$ TO $+\infty$ CONTAINS ALL THE TIMES WHEN THE PHASE STOPS CHANGING AND WE ONLY WANT ONE CYCLE. THE FUNCTION LOOKS LIKE



WE HAVE TO EVALUATE THE INTEGRAL AT $t = \pi$ NOW. TO do THAT WE EXPAND for A short PHASE ANGLE, τ, ABOUT π, I.E REPLACE t by $\pi + \tau$ AND EXPAND $\cos t = \cos(\pi + \tau) = -\cos\tau$ SINCE τ IS SMALL EXPAND THE COSINE TO first order

$$-\cos\tau = -\left(1 - \tau^2/2\right) = -1 + \tau^2/2$$

SUBSTITUTING INTO THE INTEGRAL

$$\int e^{iv\cos t}\, e^{int}\, dt = \int e^{iv(-1+\tau^2/2)}\, e^{in\pi}\, dt$$

THIS INTEGRAL CAN BE EVALUATED AND IS EQUAL TO

$$e^{-iv}\sqrt{\frac{2\pi}{v}}\; e^{i\pi/4}\; e^{in\pi}$$

NOW WE HAVE TO PUT ALL THIS STUFF TOGETHER

$$J_n(v) = \frac{1}{2\pi i^n}\left\{\sqrt{\frac{2\pi}{v}}\, e^{iv} e^{-i\pi/4} + \sqrt{\frac{2\pi}{v}}\, e^{-iv} e^{+i\pi/4}\, e^{in\pi}\right\}$$

WE CAN WRITE $\frac{1}{i^n} = e^{-in\pi/2}$, bring THIS INTO THE bracket, PULL OUT $\sqrt{2\pi/v}$ AND WE HAVE

$$J_n(v) = \frac{1}{\sqrt{2\pi v}}\left\{ e^{iv} e^{-in\pi/2} e^{-i\pi/4} + e^{-iv} e^{in\pi/2} e^{+i\pi/4}\right\}$$

THE EXPRESSION IN THE BRACKET IS JUST $2\cos\left(v - \frac{n\pi}{2} - \pi/4\right)$ SO WE HAVE FOR A FINAL EXPRESSION

$$J_n(v) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{1}{v}}\cos\left(v - \frac{n\pi}{2} - \frac{\pi}{4}\right) \qquad \text{FOR LARGE } v$$

WE should STOP HERE TO DISCUSS AGAIN THE PHYSICAL MEANING of THE SOLUTION WE have JUST obtain. IN GENERAL THE BESSEL functions APPEAR IN THE SOLUTION of A TWO DIMENSIONAL WAVE PROPAGATION Problem. WE have SOLVED for THE OUTGOING WAVE COMPONENT. THE TWO dimensional WAVE ENERGY FLOWING through EVERY CIRCLE MUST BE INVERSELY PROPORTIONAL TO THE RADIUS of THE CIRCLE. THUS THE coefficient $\sqrt{1/v}$ GIVES THE CORRECT AMPLITUDE decrement TO Produce THE correct ENERGY DECREASE AS THE WAVE PROPAGATES OUT.

The dimensionless variable v is related to the radius r as $v = \frac{\omega}{c} r$ or $v = kr$ where $k = \frac{2\pi}{\lambda}$ = wave number and λ is the wave length.

A good question to raise is how do you know when the approximation. It will turn out that the transition from small v to large v and the above asymptotic behavior occurs in the region of $v \sim n$. However, to straightforward way to go is to expand $\cos t$ to next higher order term and look at the magnitude of the correction term.

Expanding $\cos t$ to higher order we have $1 - t^2/2 + t^4/24$

Then $e^{iv\cos t} = e^{iv}\, e^{-ivt^2/2}\, e^{+ivt^4/24}\, e^{int}$. 1

Expanding the last two exponentials

$$e^{iv} \int e^{-ivt^2/2} \left(1 + \frac{ivt^4}{24} + int - \frac{n^2t^2}{2} \right) dt$$

1 gives us the previous result. int averages to zero. Now we need to evaluate $\int t^4 e^{-\alpha t^2} dt$ and $\int t^2 e^{-\alpha t^2} dt$.

Since $\int e^{-\alpha t^2} dt = \sqrt{\pi/\alpha}$ differentiating we get

$$\int t^2 e^{-\alpha t^2} dt = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \quad \text{and} \quad \int t^4 e^{-\alpha t^2} dt = \frac{3}{4} \frac{\sqrt{\pi}}{\alpha^{5/2}}.$$

Where $\alpha = iv/2$. Now the t^2 term is $1/2\alpha$ smaller than the 1st order term and t^4 is $\frac{3}{4\alpha^2}$ smaller. Collecting terms the integral

$$= e^{iv} \sqrt{\frac{2\pi}{iv}} \left[1 + \frac{iv}{24} \times \frac{3}{4(-v^2/4)} - \frac{n^2}{2} \frac{1}{2(iv/2)} \right]$$

$$= e^{iv} \sqrt{\frac{2\pi}{iv}} \left[1 + \frac{3}{24}\frac{i}{v} + \frac{in^2}{2v} \right]$$

Note that both correction terms are of the order $1/v$ and can therefore be combined. $J_n(v)$ is then given by

$$J_n(v) = \sqrt{\frac{2}{\pi}} \frac{1}{v} \cos\left(v - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \frac{1}{v^{3/2}} \text{ (sine or cosine)}$$

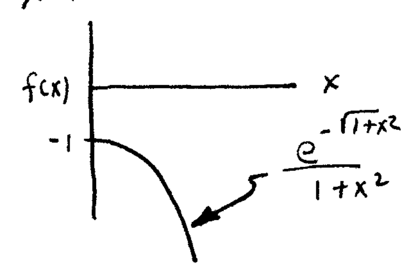
Now you have to compare terms to see if the second one is important.

YOU MAY HAVE THOUGHT IT PECULIAR THAT I KEPT TERMS IN vt^4 AND NO OTHERS EXCEPT n^2t^2. I DID THAT BECAUSE AS V IS LARGE $t \propto 1/\sqrt{V}$. THEN $Vt^4 \approx V\frac{1}{V^{4/2}} = \frac{1}{V}$ AND $n^2t^2 \propto \frac{n^2}{V}$. I HAVE INTELLIGENTLY KEPT ALL THE RIGHT TERMS.

TO SEE HOW THIS APPROXIMATION CAN WORK LET'S TAKE THE $J_0(V)$ BESSEL FUNCTION AND COMPUTE THE FIRST ROOT USING THE ABOVE APPROXIMATION. FOR THE COSINE TO BE 0 WE WANT THE ARGUMENT TO BE $\pi/2$, I.E, $V - \pi/4 = \pi/2$ OR $V = \frac{3}{4}\pi = 2.355$. THIS COMPARES WITH THE ACTUAL ROOT OF 2.405. THE NEXT ROOT REQUIRES $V - \pi/4 = 3\pi/2$, $V = 5.50$ AS COMPARED WITH 5.520. AT THE THIRD ROOT WOULD BE APPROXIMATELY 2.75π = 8.650 COMPARED WITH 8.654 ACTUALLY. ONCE THE FIRST ZERO APROXIMATION IS MADE, THE ROOTS ARE GOOD TO WITHIN A 1%.

LET'S NOW CONSIDER ANOTHER CASE OF COMPUTING HIGHLY OSCILLATORY INTEGRALS. THIS TIME WE'LL CONSIDER ONLY REAL EXPONENTS AND FURTHERMORE WE'LL SUPPOSE THE FUNCTION HAS NO MAXIMA IN THE RANGE OF INTEGRATION. HOW DO WE COMPUTE THE VALUE OF THE INTEGRAL?

CONSIDER THE INTEGRAL $\int_1^\infty \frac{e^{-(\sqrt{1+x^2})\lambda}}{1+x^2}\,dx$. IF WE PLOT THIS FUNCTION IT LOOKS LIKE:



WE WILL EVALUATE THE INTEGRAL WHERE THE FUNCTION IS BIGGEST AND THAT WILL BE AT ONE END OR THE OTHER OF INTEGRATION. IN THIS FUNCTION WE'LL LET $x \approx 1$. $e^{-\sqrt{1+x^2}}$ IS THEN $e^{-\sqrt{2}\lambda}$ AND THIS MAY BE A VERY SMALL VALUE BUT STILL ITS THE ONLY CONTRIBUTION WORTH ANYTHING.

WE WILL NOW EXPAND ABOUT $x = 1$, I.E LET $x = 1+y$ WHERE y IS A SMALL NUMBER. THE INTEGRAL BECOMES

$$\int_0^\infty \frac{e^{-\lambda\sqrt{2+2y+y^2}}}{1+(1+y)^2}\,dy$$

APPROXIMATING THE INTEGRAL

$$\int_0^\infty \frac{e^{-\lambda\sqrt{2} - \lambda Y/\sqrt{2}}}{2}\, dY = \frac{e^{-\lambda\sqrt{2}}}{2} \int_0^\infty e^{-\frac{\lambda Y}{\sqrt{2}}}\, dy$$

THE INTEGRAND IS A DYING EXPONENTIAL THAT DOESN'T REQUIRE MUCH Y TO KILL IT. SO WE HAVE AS AN APPROXIMATION

$$\int_1^\infty \frac{e^{-\lambda\sqrt{1+x^2}}}{1+x^2}\, dx \simeq \frac{e^{-\lambda\sqrt{2}}}{\sqrt{2}\,\lambda}$$

I SHOULD POINT OUT THAT $\frac{1}{\sqrt{V}}\cos(V - \frac{n\pi}{2} - \pi/4)$ IS THE STANDING WAVE SOLUTION TO THE DIFFERENTIAL EQUATION. I SHOULD HAVE TWO SOLUTIONS; THE OTHER IS THE $\frac{1}{\sqrt{V}}\sin(V - \frac{n\pi}{2} - \frac{\pi}{4})$ AND THIS SOLUTION INCLUDES SOURCES AT THE ORIGIN. THIS OTHER SOLUTION IS CALLED THE NEUMAN FUNCTION AND WE SHOULD DISCUSS IT NOW.

NEUMAN'S FUNCTIONS

FROM THE BESSEL FUNCTION DEFINITION THE DIFFERENTIAL EQUATION

$$J_n'' + \frac{1}{V} J' + (1 - n^2/V^2) J = 0$$

THE SOLUTION $\phi = A J_n(V)$ WAS OBTAINED FOR CASES OF NON SINGULARITIES AT THE ORIGIN. IF THE SOLUTION CONTAINS SINGULARITIES AT THE ORIGIN, THEN THE OTHER SOLUTION IS $B J_{-n}(V)$. THE COMPLETE SOLUTION OF DIFFERENTIAL EQUATION IS A LINEAR COMBINATION OF THE TWO SOLUTIONS, I.E,

$$\phi = A J_n(V) + B J_{-n}(V)$$

IF n IS AN INTEGER THEN $J_n(V)$ AND $J_{-n}(V)$ FUNCTIONS ARE RELATED BY THE RELATIONSHIP

$$J_n(V) = (-1)^n J_{-n}(V) \qquad \text{for } n = \text{INTEGER}$$

THE NEUMAN FUNCTION IS DEFINED IN TERMS OF THE VARIABLE p

$$N_p(V) \sin p\pi = J_p(V) \cos p\pi - J_{-p}(V)$$

FOR EVERY p NOT INTEGER.

$N_n(V)$ IS JUST THE ORTHOGONAL SOLUTION TO THE DIFFERENTIAL EQUATION. THEY OBEY THE SAME RECURSION RELATIONS AS THE $J_n(V)$'S. THE ZEROTH ORDER NEUMAN FUNCTION IS

$$N_0(V) = \ln(V)\, J_0(V) + \frac{V^2}{4} - \frac{V^2}{2^2 4^2}(1+\tfrac{1}{2}) + \frac{V^6(1+\frac{1}{2}+\frac{1}{3})}{2^2 4^2 6^2}$$

$N_0(V)$ CONTAINS A LOGARITHMIC DIVERGENCE AT $V=0$. THE TWO USEFUL RECURSION FORMULAS ARE

$$N_n' = N_{n+1} - \frac{n}{V} N_n$$

$$N_{p-1}(V)\, J_p(V) - N_p(V)\, J_{p-1}(V) = \frac{2}{\pi V}$$

THE ASYMPTOTIC SOLUTION OF $N_n(V)$ IS $\sqrt{\frac{2}{\pi V}} \sin\left(V - \frac{\pi n}{2} - \frac{\pi}{4}\right)$

A COUPLE MORE RELATIONSHIPS

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x \cosh u)\, du$$

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x \cosh u)\, du$$

HANKEL, ber, bei FUNCTIONS

THE ARE SEVERAL MORE FUNCTIONS CLOSELY RELATED TO BESSEL FUNCTIONS WHICH ARE SEEN FREQUENTLY. THEY ARE ALL FORMED BY LINEAR COMBINATIONS OF THE J'S AND N'S.

THE HANKEL FUNCTION NUMBER 1 IS DEFINED AS

1^{st} $$H^1_p(V) = J_p(V) + i N_p(V) \xrightarrow[\text{form}]{\text{ASYMPTOTIC}} \frac{e^{iV}}{\sqrt{V}}$$

2^{nd} $$H^2_p(V) = J_p(V) - i N_p(V) \longrightarrow \frac{e^{-iV}}{\sqrt{V}}$$

SOMETIMES THE BESSEL FUNCTION IS EXPRESSED IN TERMS OF A COMPLEX VARIABLE IN WHICH CASE

$$J_0(\sqrt{-i}\, x) = \mathrm{ber}\, x + i\, \mathrm{bei}\, x$$

Ber AND Bei ARE JUST THE REAL AND IMAGINARY PARTS OF THE BESSEL FUNCTION.

GENERATING FUNCTIONS

I'd like to talk about generating functions now. Generating functions are essentially sequences of functions like J_0, J_1, J_2, J_3, etc. One function contains all of the Bessel functions.

To see if we can develope a generating function for the Bessel function. Let's suppose the Bessel functions satisfy the following two relationships

$$n J_n(v) = \frac{v}{2}\left(J_{n+1} + J_{n-1}\right) \qquad (1)$$

$$J_n' = -\frac{1}{2}\left(J_{n+1} - J_{n-1}\right) \qquad (2)$$

Let $F(t,v)$ be the generating function which is defined as

$$F(t,v) = \sum_n t^n J_n(v) \qquad (3)$$

This definition says if $F(t,v)$ is expanded as power series in t then the coefficients are the Bessel functions.

Let's try and see if this definition works. Multiply eq. (1) above by t^n and sum over n, i.e

$$\sum_n n t^n J_n(v) = \frac{v}{2} t^n J_{n+1}(v) + \frac{v}{2} t^n J_{n-1}(v)$$

We need a way to express the series in terms of F. Let's (3) by t and multiply by t,

$$t \frac{d}{dt} F(t,v) = \sum n t^n J_n(v)$$

And from above

$$t \frac{\partial F}{\partial t} = \frac{v}{2}\left(\frac{1}{t} + t\right) F(t,v)$$

Proceeding to solve this differential equation

$$\frac{\partial F}{\partial t} = \frac{v}{2} F(1+t^2) \quad \rightarrow \quad \frac{dF}{F} = \frac{v}{2}(1+t^2)\,dt$$

$$= \frac{v}{2}\, d\left(t - \frac{1}{t}\right)$$

or

$$F(t,v) = A e^{\frac{v}{2}(t - 1/t)}$$

A IS A CONSTANT OF INTEGRATION AS FAR AS t GOES. A COULD BE A FUNCTION OF V. WE NEED TO SOLVE FOR $A(V)$. LET'S SOLVE $J_n' = -\frac{1}{2}(J_{n+1} - J_{n-1})$ BY MULTIPLYING BY t^n AND SUMMING

$$\sum t^n J_n' = -\tfrac{1}{2}\sum t^n J_{n+1} + \tfrac{1}{2}\sum t^n J_{n-1}$$

or

$$\frac{\partial F}{\partial V} = -\tfrac{1}{2}(1/t - t)\, F(t,V)$$

$$F(t,V) = C(t)\, e^{-V/2(1/t - t)}$$

THE CONSTANT C CAN BE A FUNCTION OF t. HOWEVER SINCE THE TWO FUNCTIONS MUST AGREE, THE SCALE HAS BEEN CHOSEN SUCH THAT $C = A = A$ AND THE FINAL RESULT IS

$$\sum_{n=-\infty}^{\infty} t^n J_n(V) = e^{V/2(t - 1/t)}$$

NOTE THAT THE POWER EXPANSION CANNOT BE IN TERMS OF $t - 1/t$ AND THIS SEEMS INCONSISTENT WITH OUR STARTING ASSUMPTION. IT IS NECESSARY TO EXPAND e^t AND $e^{-1/t}$ SEPARATELY AS THE SUMMATION RUNS FROM $-\infty$ TO $+\infty$. THE EXPANSION HAS AN ESSENTIAL SINGULARITY AT 0 AT ∞.

TO SEE IF THIS GENERATING FUNCTION WORKS LET ME EXPAND $e^{V/2(t-1/t)}$ AS A DOUBLE SERIES

$$e^{V/2(t-1/t)} = \sum_n \left(\frac{V}{2}\right)^n \frac{1}{n!} t^n \sum_k \left(-\frac{V}{2}\right)^k \frac{1}{k!}\frac{1}{t^k}$$

LET'S COMPUTE J_0, I.E. $n=0$. TO GET J_0 n MUST EQUAL k OTHERWISE ALL THE OTHER TERMS EAT EACHOTHER

$$e^{V/2(t-1/t)} = \sum_k \left(-\frac{V^2}{4}\right)^k \frac{1}{(k!)^2} = 1 - \frac{V^2}{4(1!)^2} + \left(\frac{V^2}{4}\right)^2 \frac{1}{(2!)^2} + \cdots$$

AND IT WORKS.

IT IS VERY INTERESTING THAT THE GENERATING FUNCTION CAN LOOK SO SIMPLE BUT AT THE SAME TIME REPRESENT SUCH A COMPLICATED FUNCTION. GENERATING FUNCTIONS ARE VERY USEFUL WHEN RECURSIVE RELATIONSHIPS APPEAR.

SUPPOSE $t = e^{i\theta}$ SOME COMPLEX NUMBER THEN THE GENERATING FUNCTION BECOMES

$$e^{V/2(e^{i\theta} - e^{-i\theta})} = e^{2V\sin\theta} = J_0(V) + \sum_{n>0} e^{in\theta} J_n(V) + \sum_{n>0}^{\infty} e^{-in\theta}(-1)^n J_n(V)$$

$$e^{iV\sin\theta} = J_0 + 2\sum \sin n\theta\, J_n(V)$$

THE POISSON DISTRIBUTION

- ANOTHER EXAMPLE OF A GENERATING FUNCTION -

IN OTHER TO ESTABLISH THE IDEA OF A GENERATING FUNCTION IN YOUR MIND LET ME GO THROUGH ANOTHER INTERESTING EXAMPLE. THE EXAMPLE INVOLVES THE DEVELOPMENT OF THE POISON DISTRIBUTION FUNCTION. AS YOU MAY RECALL THIS FUNCTION IS USED TO FIND THE PROBABILITY OF OBSERVING A CERTAIN NUMBER OF EVENTS IN A GIVEN TIME INTERVAL t.

THE EVENT OCCUR AT RANDOM AND ARE STATISCALLY INDEPENDENT OF EACH OTHER. WHILE IN ACTUAL PRACTICE A LOT OF EVENT DEPEND ON EACH OTHER IN SUBTLE AND COMPLICATED WAYS, WE SHALL CONSIDER THE FOLLOW EXAMPLE AS AN IDEAL ONE. SUPPOSE WE WANT TO CALCULATE: THE NUMBER OF RAIN DROPS FALLING ON A CERTAIN SQUARE FOOT OF PAVEMENT IN A GIVEN TIME INTERVAL t. WHAT IS THE PROBABILITY 57 DROPS WILL FALL IN THIS TIME INTERVAL?

WE WANT TO FIND THE PROBABILITY, $P_k(T)$, OF GETTING k COUNTS IN THE INTERVAL OF TIME T. NOW WE NEED SOME MEASURE OF THE RATE AT WHICH THE COUNTS ARE OCCURRING PER SECOND. LET US CALL THIS AVERAGE NUMBER OF COUNTS PER SECOND α. IN dT THE PROBABILITY OF AN EVENT IS PROPORTIONAL TO dT BY α. IF dT IS TOO SHORT, THEN NO EVENT WILL OCCUR. A MEASURE OF HOW LONG YOU WAIT BETWEEN EVENTS IS ON THE ORDER OF $1/\alpha$.

NOW WE WANT TO CALCULATE THE PROBABILITY OF k COUNTS OCCUR IN THE TIME INTERVAL $T + \Delta T$. THIS IS WRITTEN

$$P_k(T+\Delta T) = (1-\alpha dt)\,P_k(T) + \alpha\, dt\, P_{k-1}(T)$$

THIS IS THE SUM OF THE PROBABILITIES OF THE TWO MUTUALLY EXCLUSIVE EVENTS: THAT k EVENTS HAVE OCCURRED IN T AND NONE OCCUR IN ΔT AND THAT $k-1$ EVENTS OCCURRED IN T AND 1 EVENT IN ΔT. αdT IS THE PROBABILITY ON 1 EVENT IN dT SO $1-\alpha dT$ IS THE PROBABILITY OF NO EVENTS IN dT. $P_k(T)$ IS THE PROBABILITY YOU HAD k EVENTS IN TIME T.

BY EXPANDING THE LEFT HAND SIDE, $P_k(T+\Delta T)$ WE HAVE

$$P_k(T) + dt \frac{dP_k(T)}{dt} = P_k(T) - \alpha\, dT\, P_k(T) + \alpha\, dT\, P_{k-1}(T)$$

IN THE 0^{th} ORDER THE TWO SIDES MUST AGREE AND IN THE 1^{st} ORDER WE REQUIRE THEY AGREE, THEREFORE WE HAVE A DIFFERENTIAL EQUATION IN $P_k(T)$, VIZ

$$\frac{dP_k(T)}{dT} = -\alpha P_k(T) + \alpha P_{k-1}(T)$$

WE MUST SOLVE THIS SERIES OF DIFFERENTIAL EQUATIONS. NOTICE THE $P_k(T)$ ~~IS~~ APPEARS IN A RECURSION RELATIONSHIP WITH P_{k-1}. THUS WHENEVER THIS HAPPENS WE SHOULD CALL IN A GENERATING FUNCTION TO HELP US ALONG. WE WANT TO DEFINE A FUNCTION $F(X,t)$ WHICH HAS THE PROPERTY WHEN IT IS EXPANDED IN A POWER SERIES OF X, THE COEFFICIENTS TURN OUT TO BE THE $P_k(T)$'s. SAID MATHEMATICALLY, WE WOULD LIKE

$$F(X,T) = \sum_{k=0}^{\infty} X^k P_k(T)$$

IF WE MULTIPLY THE ABOVE EQUATION BY X^k AND SUM OVER ALL k THEN WE HAVE

$$\sum_{k=0}^{\infty} X^k \frac{dP_k}{dt} = -\alpha \sum_{0}^{\infty} X^k P_k(T) + \alpha \sum_{k=1}^{\infty} X^k P_{k-1}$$

YOU WILL NOTICE THE SECOND TERM ON THE RIGHT IS STARTED AT $k=1$ RATHER THAN $k=0$. THIS IS BECAUSE PHYSICALLY P_{-1} IS DEFINED TO BE ZERO. THE ~~MEN~~ TERM IS MEANINGLESS SO WE WILL IGNORE IT. THUS USING THE DEFINITION FOR THE GENERATING FUNCTION WE HAVE

$$\frac{dF(X,t)}{dt} = -\alpha F(X,T) + \alpha X F(X,t) = -\alpha(1-X)F$$

THIS IS EASY TO SOLVE; IT IS JUST

$$F(X,T) = C\, e^{-\alpha(1-X)T}$$

HERE AGAIN WE HAVE TO WORRY ABOUT THE CONSTANT C. THIS IS A CONSTANT WITH RESPECT TO T SO IN GENERAL IT COULD BE A FUNCTION OF X.

IN ORDER TO DETERMINE WHAT THE CONSTANT SHOULD BE LET'S EXAMINE THE SPECIAL PROPERTIES OF $F(X,T)$ WHICH ARE RELATED TO THE PHYSICS OF THE PROBLEM. CONSIDER TIME, $T=0$ THEN $F(X,0) = C(X)$. AT $T=0$ THERE IS 100% PROBABILITY OF ZERO COUNTS OCCURRING, THUS $P_0(0) = 1$. THE PROBABILITY OF 1 OR MORE EVENTS OCCURRING AT $T=0$ ARE ALL ZERO, I.E $P_1(0)=0$, $P_2(0)=0$, ETC. THEREFORE WE MUST REQUIRE THAT

$$F(X,0) = \Sigma X^k P_k(0) = 1 = C(X)$$

THUS $F(X,T) = e^{-\alpha(1-X)T}$

I WOULD NOW LIKE TO SHOW YOU HOW TO RECOVER THE $P_k(T)$'S BY EXPANDING $F(X,T)$ AS A POWER SERIES IN X. TO BEGIN LET ME WRITE

$$F(X,T) = e^{-\alpha T} e^{+\alpha XT}$$

EXPANDING THE EXPONENTIAL,

$$F(X,T) = e^{-\alpha T} \sum_{k=0}^{\infty} \frac{(\alpha XT)^k}{k!}$$

THEREFORE WE HAVE DEDUCED

$$P_k(T) = \frac{(\alpha T)^k e^{-\alpha T}}{k!}$$

I'D NOW LIKE TO TALK ABOUT THE PHYSICS OF THIS FUNCTION. FIRST IT SHOULD BE TRUE THAT THE PROBABILITY OF ALL EVENTS SHOULD ADD UP TO 1, I.E

$$\sum_{k=0}^{\infty} P_k(T) = \sum_{k=0}^{\infty} \frac{(\alpha T)^k}{k!} e^{-\alpha T} = e^{+\alpha T} e^{-\alpha T} = 1$$

AND IT DOES. NOW TO OBTAIN THE MEAN NUMBER OF COUNTS, N, I MUST COMPUTE $\Sigma k P_k(T)$, I.E

$$N = \Sigma k P_k(T) = \Sigma k \frac{(\alpha T)^k}{k!} e^{-\alpha T}$$

SINCE $\frac{k}{k!} = \frac{1}{(k-1)!}$ WE HAVE $\Sigma \frac{(\alpha T)^k}{(k-1)!} e^{-\alpha T} = \Sigma (\alpha T) \frac{(\alpha T)^{k-1}}{(k-1)!} e^{-\alpha T}$

AND BY REDEFINING $k-1 = l$ AND SUMMING WE HAVE

$$N = \alpha T$$

BEFORE I GO ON I WOULD LIKE TO POINT OUT SOME UTILITY OF THE GENERATING FUNCTION IN UNDERSTANDING THE RESULTS. LET'S REWRITE $F(X,T) = e^{-\alpha(1-X)T} = \sum X^k P_k(T)$. IF THIS WERE NOT SUCH AN EASY FUNCTION TO EXPAND IN TERMS OF X WE COULD STILL CHECK TO SEE IF IT IS CORRECT. E.G. THE PROBABILTY OF 100% IS JUST $X = 1$ OR $F(1,T) = e^{-0} = 1$. THE MEAN CAN BE COMPUTED FROM DIFFERENTIATING F WITH RESPECT TO X AND EVALUATING AT $X = 1$

$$\left.\frac{\partial F(X,T)}{\partial X}\right|_{X=1} = \sum k X^{k-1} P_k = \sum k \frac{X^k}{X} P_k$$
$$= \alpha T \, e^{-\alpha(1-X)T} = \alpha T = N$$

TO COMPUTE THE MEAN SQUARE OF THE NUMBER WE WANT

$$\sum k^2 P_k(T) = \text{MEAN SQUARE}$$

IT IS EASIER TO FIRST COMPUTE THE MEAN SQUARE MINUS THE MEAN SINCE

$$\left.\frac{\partial^2 F}{\partial X^2}\right|_{X=1} = \sum k(k-1) P_k(T) = (\alpha T)^2 + \alpha T = N^2 + N$$

NOW IF THE MEAN NUMBER OF COUNTS IS N THEN IN A UNIT TIME INTERVAL WE EXPECT TO GET

$$\frac{N^k e^{-N}}{k!}$$

THE PROBABILITY OF GETTING NO COUNTS IS JUST e^{-N}. THUS THE PROBABILITY DIES EXPONENTIALLY AS N. AT THE OTHER EXTREME FOR VERY LARGE N THE PROBABILITY DISTRIBUTION CAN BE APPROXIMATED USING THE FACT THAT

$$k! \approx \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \qquad \text{for LARGE } k$$

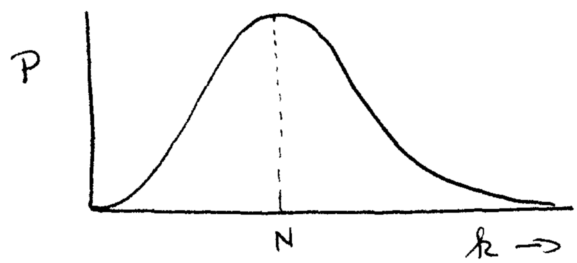
k IS ASSUMED LARGE SINCE N IS LARGE IMPLYING A LOT OF EVENTS OCCURRING. THEN WE HAVE

$$P_k = \frac{N^k e^{-N}}{k!} \approx \frac{1}{\sqrt{2\pi k}} \left(\frac{Ne}{k}\right)^k e^{-N} = \frac{1}{\sqrt{2\pi k}} \left(\frac{N}{k}\right)^k e^{-N+k}$$

I CAN WRITE THIS AS

$$P = \frac{1}{\sqrt{2\pi k}}\, e^{-N + k + k \ln\left(\frac{N}{k}\right)}$$

THIS CURVE LOOKS LIKE



THE MAXIMA OCCURS AT $k = N$. I WILL EXPAND P ABOUT N BY LETTING $k = N + l$ WHERE l IS A + or - SMALL NUMBER,

$$P_k = \frac{1}{\sqrt{2\pi N}}\, e^{+l + (N+l)\ln\left(\frac{N}{N+l}\right)}$$

Now $\ln \dfrac{N}{N+l} = -\ln\left(1 + \frac{l}{N}\right) \simeq -\dfrac{l}{N} + \dfrac{1}{2}\dfrac{l^2}{N^2}$ + higher order

SUBSTITUTING BACK IN

$$P_k = \frac{1}{\sqrt{2\pi N}}\, e^{l + (N+l)\left(-\frac{l}{N} + \frac{1}{2}\frac{l^2}{N^2}\right)} \simeq \frac{1}{\sqrt{2\pi N}}\, e^{-l^2/2N}$$

THUS P_k IS APPROXIMATED AS A GAUSSIAN NEAR THE MEAN

$$P_k \approx \frac{1}{\sqrt{2\pi N}}\, e^{-\frac{(N-k)^2}{2N}}$$

THE WIDTH OF THE GAUSSIAN IS PROPORTIONAL TO $\sqrt{N}$.

GAMMA FUNCTION

I NOW WANT TO TALK ABOUT ANOTHER SPECIAL FUNCTION WHICH IS ENCOUNTERED FREQUENTLY – THE GAMMA FUNCTION. THE GAMMA FUNCTION IS COMMONLY USED WHEN NONINTEGRAL FACTORIALS ARE ENCOUNTERED. FACTORIALS ARE MORE CUSTOMARIALLY DEFINED FOR INTEGER VALUES OF n. THE FOLLOWING DEFINITION OF THE GAMMA FUNCTION IS ADOPTED

$$\Gamma(n+1) = n! = \int_0^\infty x^n e^{-x}\,dx \qquad n > 0$$

THE INTEGRAL IS NOT DEFINED FOR $n < 0$ SO THE ABOVE DEFINITION APPLIES ONLY FOR $n > 0$.

TO PROVE THE INTEGRAL RESULTS IN THE FACTORIAL RECALL THAT $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$. IF I BEGIN TO DIFFERENTIATE WITH RESPECT TO x I WILL START TO GET THE FOLLOWING INTEGRALS

$$\int_0^\infty x\, e^{-\alpha x}\,dx = \frac{1}{\alpha^2} \quad , \quad \int x^2 e^{-\alpha x}\,dx = \frac{1\cdot 2}{x^3} \quad , \quad \int x^3 e^{-\alpha x}\,dx = \frac{3\cdot 2\cdot 1}{\alpha^4}$$

AND WITH LITTLE MENTAL EFFORT THIS CAN BE EXTENDED TO

$\int x^n e^{-\alpha x}\,dx = \dfrac{n!}{\alpha^{n+1}}$ WHICH FOR $\alpha = 1$ IS JUST THE ABOVE FUNCTION.

IN ORDER TO EVALUATE $\frac{1}{2}!$ WE NEED TO EXTEND THE DEFINITION OF THE GAMMA FUNCTION NOTING THAT FROM THE DEFINITION $n! = F(n+1)$ CAN BE REWRITTEN USING THE FACT THAT $n! = n(n-1)!$ THE RESULTING RECURSION RELATIONSHIP IS DEVELOPED

$$F(z+1) = z\,F(z)$$

HERE z CAN BE INTEGER, NONTEGER, REAL OR COMPLEX. THUS FROM THE DEFINITION WE HAVE AS THE FIRST FEW TERMS

$$\Gamma(1) = 0! = 1 \qquad \Gamma(2) = 1! = 1 \qquad \Gamma(3) = 2! = 2 \text{ , ETC.}$$

NOW LET'S TRY TO FIND $\Gamma(\frac{1}{2})$. SUBSTITUTING INTO THE INTEGRAL

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-1/2} e^{-x}\,dx$$

BY SUBSTITUTING $x = y^2$ THE INTEGRAL BECOMES $\int_0^\infty 2e^{-y^2}\,dy = \sqrt{\pi}$

THUS WE HAVE

$$(+\tfrac{1}{2})! = \sqrt{\pi}$$

USING THE RECURSION RELATIONSHIP WE HAVE THAT $\Gamma(3/2) = 1/2\,\Gamma(1/2) = 1/2\sqrt{\pi}$. THUS IN GENERAL

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-3/2)(n-5/2)\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$
$$= \frac{(2n-1)(2n-3)(2n-5)\cdots\cdots 5\cdot 3\cdot 1}{2^n}\sqrt{\pi}$$

OR THE PRODUCT OF ALL THE ODD INTEGERS IS JUST GIVEN BY THE FACTORIAL FUNCTION

$$\Gamma(n+\tfrac{1}{2}) = \frac{2n!}{2^{2n}\,n!}\sqrt{\pi}$$

THE BESSEL FUNCTION CAN BE DEFINED IN TERMS OF THE THE GAMMA FUNCTION AS

$$J_n(z) = \frac{2}{\sqrt{\pi}}\frac{(z/2)^n}{\Gamma(n+1/2)}\int_0^{\pi/2}\cos(z\cos\varphi)\sin^{2n}\varphi\,d\varphi$$

WHEN N IS NON INTEGER THE BESSEL FUNCTION IS STILL DEFINED IN TERMS OF THE GAMMA FUNCTION. HOWEVER IF N RESULTS IN INTEGRATING OVER ONLY PART OF A CYCLE, THE PROBLEM FALLS APART AND YOU HAVE NO TASTE FOR PICKING A BAD N. IN THE CASE OF $J_{1/2}(z)$ THE ABOVE BECOMES

$$J_{1/2}(z) = \frac{2z}{z\sqrt{\pi}}\cancel{\sin(z)}\left[1-\cos z\right]$$

THE BETA FUNCTION

SOMEONE HAS BOTHERED TO DEFINE A NEW FUNCTION CALLED THE BETA FUNCTION TO DEAL WITH MULTIPLYING TWO GAMMA FUNCTIONS TOGETHER. LET'S START BY MULTIPLYING $\Gamma(m)$ BY $\Gamma(n)$

$$\Gamma(m)\Gamma(n) = \int_0^\infty\int_0^\infty x^{m-1}e^{-x}\,dx\,Y^{n-1}e^{-Y}\,dY$$

NOW DEFINE $X = \xi^2$ AND $Y = \eta^2$ THEN

$$\Gamma(m)\Gamma(n) = \int_0^\infty\int_0^\infty \xi^{(2m-1)}\eta^{(2n-1)}e^{-(\xi^2+\eta^2)}\,d\xi\,d\eta$$

IF WE THINK OF ξ AND η AS DEFINING A PLANE, WE WANT TO COMPUTE AN AREA INTEGRAL; SO AGAIN LET'S CHANGE VARIABLES

$$\xi = r\sin\theta \quad \text{AND} \quad \eta = r\cos\theta$$

Thus we have

$$\Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r^{(2m-1+2n-1)}\, r\,dr\, \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Now the integral over the radius r is just another Gamma function, $\Gamma(m+n)$. Therefore

$$\Gamma(m)\Gamma(n) = 2\,\Gamma(m+n)\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

or dividing out $\Gamma(m+n)$ on the right

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

This integral which equals $\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ is the definition of the Beta function of m and n, i.e.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

It can be shown that

$$B(m,n) = \int_0^1 (1-x)^{n-1}\, x^{m-1}\, dx$$

It is true that $B(m,n) = B(n,m)$. All it can be shown that

$$B(m,n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

In the special case $m+n=1$ then

$$\Gamma(m)\Gamma(1-m) = \int_0^{\infty} \frac{y^{m-1}}{1+y}\, dy$$

We had trouble trying to compute this integral by contour integration methods but it is straightforward to evaluate via Gamma functions that

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin \pi m}$$

For $m = 1/2$ $\quad [\Gamma(1/2)]^2 = \pi$ $\quad$ such that $\quad \Gamma(1/2) = \sqrt{\pi}$

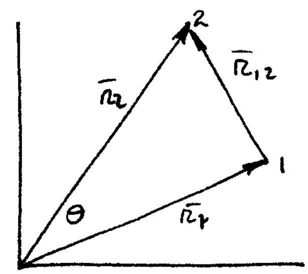
LEGENDRE POLYNOMIALS

THERE IS ONE MORE CLASS OF SPECIAL FUNCTIONS THAT HAS BEEN DIGNIFIED BY A NAME SO NOW I WANT TO DISCUSS THEM. THESE ARE THE SO CALLED ~~BO~~ LEGENDRE POLYNOMIALS. NOW THERE ARE MANY WAYS TO DEFINE THIS SET OF FUNCTIONS BUT I'D LIKE TO DO IT VIA THE GENERATING FUNCTION METHOD PREVIOUSLY DISCUSSED.

THE GENERATING FUNCTION FOR THE LEGENDRE POLYNOMIALS IS

$$F(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

NOW IN ADDITION TO HAVING A NUMBER OF MATHEMATICAL IMPLICATION THE GENERATING FUNCTION HAS AN IMPORTANT GEOMETRICAL SIGNIFICANCE WHICH I WANT TO MENTION. IF I WANT TO SPECIFY THE DISTANCE BETWEEN 2 POINTS IN POLAR COORDINATES I WOULD WRITE



$$r_{12} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}$$

IF I CONSIDER $r_1 < r_2$ FIRST, I CAN AS THE RECIPROCAL DISTANCE

$$\frac{1}{r_{12}} = \frac{1}{r_2\sqrt{\left(\frac{r_1}{r_2}\right)^2 + 1 - 2\frac{r_1}{r_2}\cos\theta}}$$

THE SQUARE ROOT CAN BE EXPANDED AS A POWER SERIES

$$\frac{1}{r_{12}} = \frac{1}{r_2}\sum \left(\frac{r_1}{r_2}\right)^n P_n(\cos\theta)$$

THUS WE CAN EQUATE THE VARIABLES IN THE GENERATING FUNCTION IN THIS EXPRESSION

$$t = \frac{r_1}{r_2}, \qquad x = \cos\theta$$

SINCE IN ELECTROSTATICS THE POTENTIAL ENERGY IS INVERSELY PROPORTIONAL TO THE DISTANCE BETWEEN THE CHARGES, I.E. $V = K/r_{12}$ OR

$$V = \frac{K}{r_2}\sum \left(\frac{r_1}{r_2}\right)^n P_n(\cos\theta)$$

THUS A NUMBER OF PROBLEMS COME UP WHERE THIS GENERATING FUNCTION APPEARS SO THE LEGENDRE POLYNOMIALS ARE USEFUL TO KNOW.

Now back to the generating function and to discover something about these special functions. Let's expand $(1-2tx+t^2)^{-1/2}$ in a power series

$$(1-2tx+t^2)^{-1/2} = 1 - \tfrac{1}{2}(-2tx+t^2) - \tfrac{3}{8}(-2tx+t^2)^2 + \text{higher order}$$

collecting terms in orders of t,

$$(1-2tx+t^2)^{-1/2} = 1 + tx + \left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right)t^2 + \text{higher order}$$

Thus we have the first few Legendre polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$$

Now the higher order Legendre polynomials can be computed using the following recursion relationship,

$$(n+1)\,P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0$$

To show how this works let's compute P_3 from the above functions

$$3P_3 - 5x\,P_2 + 2P_1 = 0$$
$$3P_3 = 5x\,P_2 - 2P_1 = 5x\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) - 2x$$
$$= \tfrac{15}{2}x^3 - \tfrac{9}{2}x$$

or

$$P_3 = \tfrac{5}{2}x^3 - \tfrac{3}{2}x$$

Another useful recursion relationship involves the derivative of the Legendre polynomial, viz,

$$(1-x^2)\,P_n'(x) + n\,x\,P_n(x) - n\,P_{n-1}(x) = 0$$

Using the generating function to compute integrals

I want to show you how to use the generating function to compute the values of integrals. Suppose I want to evaluate

$$I_{mn} = \int_0^{\pi} P_m(\cos\theta)\,P_n(\cos\theta)\,\sin\theta\,d\theta$$

This is not one integral but a whole set of integrals. How do I do that? Well, if I redefine the variables I have

$$I_{mn} = \int_{-1}^{+1} P_m(x)\,P_m(x)\,dx$$

LET ME DEFINE A NEW GENERATING FUNCTION $G(t,s)$ WHICH HAS THE FOLLOWING DEFINITION

$$G(t,s) = \sum_m \sum_n I_{mn} t^m s^n$$

THIS PERMITS DOING THE WHOLE SET OF INTEGRALS AT ONCE FOR ALL I HAVE TO DO IS EXPAND $G(t,s)$ TO GET THE VALUES OF THE INTEGRALS. FIRST, HOWEVER, I MUST INTEGRATE

$$G(t,s) = \sum_m \sum_n \int_{-1}^{+1} t^m P_m(x)\, s^n P_n(x)\, dx$$

SUBSTITUTING IN FOR THE GENERATING FUNCTIONS OF $P_n(x)$

$$G(t,s) = \sum_m \sum_n \int_{-1}^{+1} \frac{1}{\sqrt{1-2tx+t^2}} \, \frac{1}{\sqrt{1-2sx+s^2}}\, dx$$

IF I CAN DO THE INTEGRAL ON X, I CAN GET THE ANSWER I AM AFTER. UNFORTUNATELY, I HAVE CONSTRUCTED AN INTEGRAL WHICH IS HARDER TO EVALUATE THAN THE ORIGINAL SET. THIS IS NOT ALWAYS TRUE SO DON'T BE DISCOURAGED. IT DOES TURN OUT THAT THE ABOVE INTEGRAL CAN BE EVALUATED AS AN INDEFINITE INTEGRAL AND IT CAN BE SIMPLIFIED TO

$$G(t,s) = \frac{1}{\sqrt{st}} \ln \frac{1+\sqrt{st}}{1-\sqrt{st}} = \frac{1}{\sqrt{w}} \ln \frac{1+w}{1-w}$$

I HAVE TO EXPAND THIS AS A POWER SERIES IN W

$$G(t,s) = \frac{1}{w}\left[\begin{array}{l} +w - w^2/2 + w^3/3 - w^4/4 + \cdots \\ +w + w^2/2 + w^3/3 + w^4/4 + \cdots \end{array}\right.$$

$$= 2\left[1 + \frac{w^2}{3} + \frac{w^4}{5} + \frac{w^6}{7} + \frac{w^{2k}}{2k+1}\right.$$

$$= 2 \sum_k \frac{\left(\sqrt{st}\right)^{2k}}{(2k+1)} = 2\sum_{k=0}^{\infty} \frac{s^k t^k}{2k+1}$$

THUS WE SEE THAT

$$I_{mn} = \frac{2}{2k+1}$$

WHAT DOES THIS MEAN? FIRST IT SAYS THAT IF m DOES NOT EQUAL n THEN $I_{mm} = 0$, OTHERWISE IF $m = n = k$, THEN

$$I_{mn} = \frac{2}{2k+1}$$

AND THE INTEGRAL HAS THIS VALUE.

The fact that $I_{mn} = 0$ for $m \neq n$ is a very important result since it implies that the two functions are orthogonal. This property is a backbone of a good part of mathematics and it is central to the definition of a complete set of functions like the ~~generatin~~ Legendre polynomials

Through the definition of $P_n(x)$ and the recussion relationship it can be shown that $P_n(x)$ satisfies the following differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + n(n+1)P_n(x) = 0$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}\right]P_n(\cos\theta) + n(n+1)P_n(\cos\theta) = 0$$

The reason I draw attention to this equality is that within the realm of physics this differential equation comes up very frequently and we need to solve it. The reason this equation comes up often is that many problems involve the Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This operator works on some function $f(x,y,z)$. If we transform from Cartesian coordinates to spherical coordinates then the Laplacian operator must likewise be transformed. We must use the relationships

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

After doing the algebra

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial F}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}$$

Sometimes it is more convenient to the radial term as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rF)$$

You can show that indeed

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rF) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)$$

LET'S NOW DISCUSS A PROBLEM INVOLVING THE WAVE EQUATION IN SPHERICAL COORDINATES, I.E $\nabla^2 F = -k^2 F$. SUPPOSE FIRST THERE IS NO DEPENDENCE ON φ; THIS MAKES THE PROBLEM ONE DEGREE LESS COMPLICATED. SINCE WE KNOW $P_n(\cos\theta)$ SATISFIES THE θ DEPENDENCE, LET'S TRY THE SOLUTION

$$F = f_n(r)\, P_n(\cos\theta)$$

THIS IS AN OUTGOING SPHERICAL WAVE WITH SOME ANGULAR DISTRIBUTION DESCRIBED BY $P_n(\cos\theta)$. IF I PLUG THIS TRIAL SOLUTION INTO THE DIFFERENTIAL EQUATION, I GET

$$Y(\theta)\,\frac{1}{r}\frac{d^2}{dr^2}\left(r f_n(r)\right) + \frac{f_n(r)}{r^2\sin\theta}\frac{d}{d\theta}\sin\theta\,\frac{dY(\theta)}{d\theta} = -k^2 f Y$$

WHERE $Y(\theta) = P_n(\cos\theta)$. BY DIVIDING THROUGH BY fY AND MULTIPLYING THROUGH BY r^2 WE UNCOUPLE THE RADIAL AND ANGULAR PARTS. THUS $Y(\theta)$ MUST SATISFY

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\,\frac{dY}{d\theta} + n(n+1)Y = 0$$

THUS $Y(\theta)$ IS THE VERY SPECIAL FUNCTION $P_n(\cos\theta)$.

NOW THE RADIAL PART MUST SATISFY

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r f_n(r)\right) + \frac{n(n+1)}{r^2} f_n(r) = -k^2 f_n(r)$$

THE SOLUTION TO THIS DIFFERENTIAL EQUATION IS

$$f_n(r) = A\sqrt{\frac{\pi}{2kr}}\, J_{n+1/2}(kr) + B\sqrt{\frac{\pi}{2kr}}\, N_{n+\frac{1}{2}}(kr)$$

I SHOULD MENTION THERE IS ANOTHER SET OF FUNCTIONS WHICH ARE RELATED TO THE P_n'S AS

$$Q_n(x) = \frac{1}{2}\int_{-1}^{+1} P_n(y)\,\frac{dy}{x-y}$$

SINCE THERE IS A POLE AT $x=y$ WE NEED THE PRINCIPAL VALUE OF THE INTEGRAL. A WORD ABOUT THE PRINCIPAL VALUE OF AN INTEGRAL IS IN ORDER. SUPPOSE I WANT TO CALCULATE $Q_0(x)$, THEN I WANT TO EVALUATE

$$Q_0(x) = \frac{1}{2}\int_{-1}^{+1}\frac{1}{x-y}\,dy$$

THIS INTEGRAL BLOWS UP AT $Y = X$ SO LET ME BACK OFF ON EITHER SIDE OF THE POLE BY AN AMOUNT ϵ AND TAKE THE SUM OF THE TWO PARTS. I.E, GET RID OF ~~OF~~ THE TROUBLE SPOT BY LETTING ϵ GOR TO 0 IN THE LIMIT

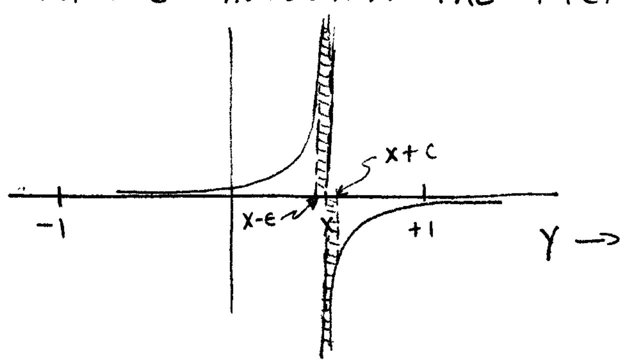
$$Q_0 = \lim_{\epsilon \to 0} \frac{1}{2} \left[\int_{-1}^{x-\epsilon} \frac{dy}{x-y} + \int_{x+c}^{+1} \frac{dy}{x-y} \right]$$

$$= \frac{1}{2} \left\{ \left[-\ln(x-y) \right]_{-1}^{x-\epsilon} + \left[-\ln(x-y) \right]_{x+c}^{1} \right\}$$

$$= \frac{1}{2} \left[-\cancel{\ln \epsilon} + \ln(x+1) - \ln(x-1) + \cancel{\ln \epsilon} \right]$$

$$= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

NOTICE THE DEPENDENCE ON ϵ DROPS OUT AS IT SHOULD IF WE TAKE EQUALS AMOUNTS ON EITHER SIDE OF X. THUS WE HAVE A LARGE POSITIVE AREA BEING EATEN BY AN EQUAL BUT NEGATIVE AMOUNT. THE PICTURE LOOKS LIKE



IT TURNS OUT ALL THE Q'S CONTAIN LOGARITHMIC TERMS PLUS OTHER POLYNOMIALS. THE Q'S ARE USEFUL TO PHYSICS AS THEY COME IN WHEN A WAVE IS GENERATED FROM A SOURCE ALONG THE z - AXIS.

I HAVE NOT MENTIONED THE OTHER SOLUTION TO THE RADIAL EQUATION, THIS SOLUTION IS ASSOCIATED WITH EXPANDING FOR $r_2 < r_1$. IN THIS CASE $f_n(r) = \frac{1}{r^{n+1}}$ AND $F = \frac{1}{r^{n+1}} P_n(\cos\theta)$. IS A SOLUTION TO THE DIFFERENTIAL EQUATION. THIS SOLUTION OBTAINS WHEN THERE ARE CHARGES CONCENTRATED AT THE ORIGIN. FOR 1 CHARGE $n = 0$, $P_0 = 1$ AND $F = 1/r$. THE SPHERICAL WAVE IS SYMMETRIC SPHERICALLY.

For two charges at the origin a − and + charge a dipole is formed with a $1/r^2$ electric potential variation. The angular distribution goes as $P_1(\cos\theta) = \cos\theta$. The next possibility if $n=2$ or 3 charges at the origin; two pluses and 1 minus. This has a more complicated electric potential; it falls off as $1/r^3$ and angularly varies as $\frac{1}{2}(3\cos^2\theta - 1)$

φ Angular Dependence

So far we have not worried about any φ or azimuthal angular dependence what if the distribution depends on φ? Then the differential equation becomes complicated by $\frac{1}{r^2\sin^2\theta}\frac{d^2F}{d\varphi^2}$. However, this is the only term which depends on φ so we know F must be proportional to itself, i.e $F = A\sin m\varphi + B\cos m\varphi$. Thus the more general solution for F is

$$F = \sqrt{\frac{\pi}{2kr}}\; J_{n+\frac{1}{2}}(kr)\, P_n^m(\theta)\sin m\varphi$$

$P_n^m(\theta)$ is the analog of the Legendre polynomial for other cases of $m \neq 0$ and is related to $P_n(\cos\theta)$ by

$$P_n^m(\cos\theta) = (\sin\theta)^m \frac{d^m}{dx^m} P_n(x)$$

$P_n^m(\cos\theta)$ is called the associated Legendre functions.

Let's look at the angular distribution for some special cases. First for $n=0$ $P_0(\cos\theta) = 1$ This is the trivial case; there is no angular dependence. Next $n=1$ contains three possibilities $m=0$ or 1 but m must always be less than or equal to n. Thus we have

$$\begin{array}{llll} n=1 & m=0 & P_1^0(\cos\theta) = \cos\theta & = z/r \\ n=1 & m=1 & P_1^1(\cos\theta) = \sin\theta\cos\varphi & = x/r \\ & & \qquad\qquad = \sin\theta\sin\varphi & = y/r \end{array}$$

These three functions are unique in that they contain no dependence on the definition of the x, y, z coordinate system. That is any transformation can be expressed as a linear combination of the above three functions.

Now lets look at $n=2$ This time there are five possibilities $m = 0, \pm 1, \pm 2$

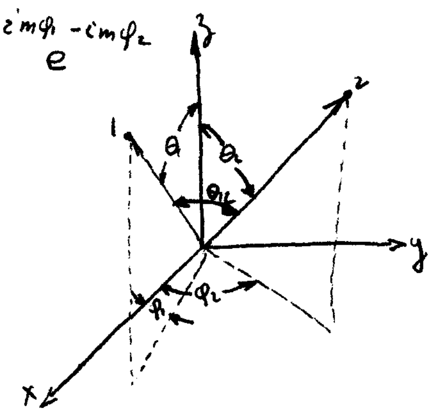
P_0^2	$\longrightarrow$	$\frac{3}{2}\cos^2\theta - \frac{1}{2}$	$\frac{3z^2 - r^2}{2r^2}$
P_1^2		$3\cos\theta\sin\theta\cos\varphi$	zx/r^2
P_1^2		$3\cos\theta\sin\theta\sin\varphi$	zy/r^2
P_2^2		$3\sin^2\theta\cos 2\varphi$	$(x^2 - y^2)/r^2$
P_2^2		$3\sin^2\theta\sin 2\varphi$	$2xy/r^2$

There is another useful relationship between two points defined in spherical coordinates that should be mentioned. The included angle between the two points, θ_{12}, is given by

$$\cos\theta_{12} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2)$$

This can be expressed as

$$P_n(\cos\theta_{12}) = \sum \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_1)\, P_n^m(\cos\theta_2)\, e^{im\varphi_1} e^{-im\varphi_2}$$



FOURIER SERIES

I now want to introduce the subject of Fourier series by discussing a problem which Fourier himself attempted to solve and in the process developed the very useful and general concept of Fourier series. The problem dealt with a uniform rod of length a. The question was to determine at some later time t what the temperature would be at some position X - assuming the initial temperature distribution was know at $t=0$.

The problem required solving the differential equation

$$\nabla^2 T = + \frac{\partial T}{\partial t}$$

which simplifies to $\frac{\partial^2 T}{\partial x^2} = + \frac{\partial T(x,t)}{\partial t}$ if we consider a one dimensional rod. This differential equation has a special solution given by $T(x,t) = f(x)\, g(t)$ which we can try by plugging into the equation. If we do we get

$$f''(x)\, g(t) = + f(x)\, g'(t)$$

Dividing through by $f(x)\, g(t)$ we have

$$\frac{f''(x)}{f(x)} = + \frac{g'(t)}{g(t)} = \text{constant}$$

Since the equation in f does not depend on t we must require that $f''(x)/f(x) = \text{constant}$ or that $f(x)$ be exponential. If we take the constant to be $-k^2$ then

$$f(x) = A \sin kx$$

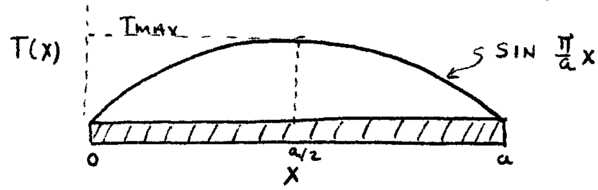
In general $f(x) = A\sin kx + B\cos kx$ but I am requiring $T = 0$ at $x = a$ and $x = 0$ Thus $\cos kx$ cannot be a solution at $x=0$. At $x = a$ we want $f(a) = 0$ thus $\sin ka = 0$ which requires that $k = \frac{n\pi}{a}$. Solving now for $g(t)$ we find $g(t) = e^{-k^2 t}$

The solution for $T(x,t)$ is then

$$T(x,t) = A \sin\left(\frac{n\pi}{a} x\right) e^{-\frac{n^2\pi^2}{a^2} t}$$

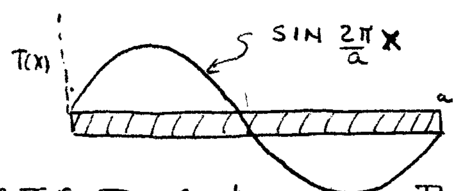
A is a constant which hasn't been defined yet.

Let's look at this solution for a couple of values of n and see if we can understand what it tells us. First let $n = 1$ at $t = 0$. Then the rod has a temperature profile that looks like the following:



As time advances this temperature profile will collapse exponentially, i.e., all amplitudes along X go down proportionately with a time constant, τ given by a^2/π^2. τ is the time to reduce the amplitude $1/e$, or .368 of the original value. Note also the bigger a the slower the temperature falls off.

Now let $n = 2$ then at $T(x,t) = T(x,0)$ the distribution looks like

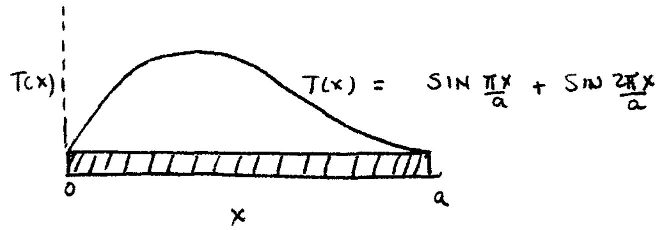


Again at a later time t this distribution is exponentially decay but with 4 times the rate of the $n = 1$ distribution. τ now becomes $a^2/4\pi^2$.

What would happen if the initial temperature distribution was not a simple sinusoid? How would we figure out what the temperature is at a later time t? This is what Fourier wanted to know. He found out that it wasn't necessary to resolve the problem over and over again. To understand what he did we must make use of one important property of the linear differential equation $\nabla^2 T = \frac{\partial T}{\partial t}$. That is if $T_1(x,t)$ and $T_2(x,t)$ are solutions then a linear combination of the two is also a solution, that is,

$$T_3(x,t) = \alpha T_1(x,t) + \beta T_2(x,t)$$

Thus we can now add the $n = 1$ and $n = 2$ solutions given above to find a new solution which has an initial temperature distribution that looks like,



NOW AT A LATER TIME t THIS DISTRIBUTION BECOMES MORE AND MORE LIKE THE $n=1$ DISTRIBUTION BECAUSE THE $n=2$ COMPONENT IS DYING OFF 4 TIMES FASTER THAN THE $n=1$ PART. THUS AS TIME PASS THE LOWEST ORDER ~~SURVI~~ TERM SURVIVES, THE DISTRIBUTION LOOKS MORE SYMMETRICAL AND DIES OFF WITH THE TIME CONSTANT OF $n=1$.

FOURIER CONCLUDED THAT ANY SMOOTH TEMPERATURE DISTRIBUTION COULD BE WRITTEN AS THE SUM OF A SERIES OF SINUSOIDAL COMPONENTS OF DIFFERENT AMPLITUDES AND DIFFERENT MODE SHAPES, I.E.

$$T(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a}\, e^{-\frac{\pi^2 n^2}{a^2} t}$$

IF THE INITIAL TEMPERATURE DISTRIBUTION IS KNOW, I.E $f(x) = T(x,0)$ THEN IT CAN BE EXPRESSED AS

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

WHAT KIND OF FUNCTIONS HAVE THIS SPECIAL PROPERTY OF BEING EXPRESSIBLE AS THE SUM OF SERIES OF SINES AND COSINES. ALSO HOW DO WE COMPUTE THE A_n'S, I.E THE CONTRIBUTION OF EACH MODE?

FOURIER NOTICED THAT THE A_n'S COULD BE RETRIEVED BY THE FOLLOWING METHOD

$$\int_0^a f(x) \sin \frac{m\pi x}{a}\, dx = \sum A_n \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx$$

THE INTEGRAL ON THE RHS HAS THE WONDERFUL PROPERTY OF BEING 0 IF $m \neq n$ AND IS $a/2$ IF $m = n$. THE PROOF FOR THIS IS STRAIGHT FORWARD

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \int \frac{1}{2}\left[\cos \frac{(n-m)\pi x}{a} - \cos \frac{(n+m)\pi x}{a}\right] dx$$

$$= \frac{1}{2}\left\{ \frac{a}{\pi(m-n)} \sin \frac{(n-m)\pi x}{a} - \frac{a}{\pi(m+n)} \sin \frac{(n+m)\pi x}{a} \right]_0^a$$

$$= \quad 0 - 0 = 0 \quad \text{for } m \neq n$$

This if $m=n$ the integral is not defined so we must redo if

$$\int \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a}\, dx \overset{m=n}{=} \int_0^a \frac{1}{2}\cos\frac{2m\pi x}{a}\, dx = \frac{a}{2}$$

Thus we have that

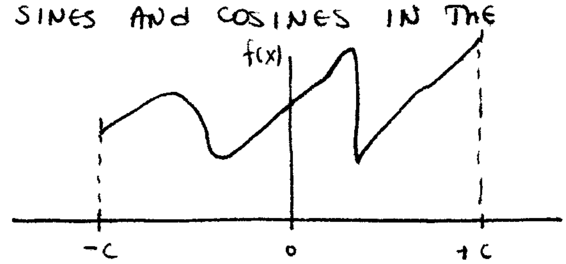
$$A_m = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

The A_m's are called the coefficients of the fourier series.

Not that the coefficients are found we must find the f(x)'s. Essentially all smooth, singled value functions are okay. That lacks mathematical rigour but it works. The more terms that are taken in the series the better the fit to the actual function.

FOURIER'S THEOREM.

I want to return to a more general approach now of fourier series. I have some periodic function, f(x) and I want to express it in terms of sines and cosines in the interval $-c$ to $+c$. Over this interval f(x) can be expressed as



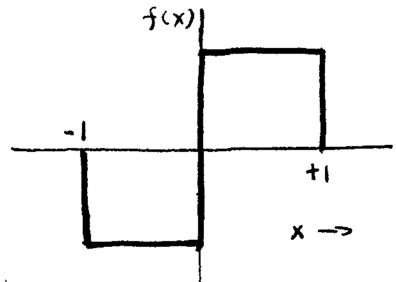
$$f(x) = \frac{1}{2}B_0 + B_1\cos\frac{\pi x}{c} + B_2\cos\frac{2\pi x}{c} + \cdots + A_1\sin\frac{\pi x}{c} + A_2\sin\frac{2\pi x}{c} + \cdots$$

Where the sine and cosine coefficients are given by

$$A_N = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\, dx \qquad B_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\, dx$$

This is called Fourier's Theorem of inverting a series. The mathematicians found it hard to believe when it was first proposed. As long as f(x) has a finite number of discontinuities and is singled value, this series expansion is valid. Note that f(x) repeats every 2c. Furthermore if f(x) is a symmetrical function about $x=0$ only the cosine terms remain. If F(x) is antisymmetrical only the sine terms remain.

LET'S WORK OUT AN EXAMPLE of HOW A FUNCTION IS EXPRESSED IN TERMS of A FOURIER SERIES. LET $f(x)$ BE THE FOLLOWING



$f(x) = -1$ for $x < 0$

$f(x) = +1$ for $x > 0$

FIRST WE'LL FIND THE A_N'S; NOTICE THE FUNCTION IS ANTI-SYMMETRIC SO ONLY THE SINE TERMS SURVIVE. THUS

$$A_N = \frac{1}{c}\int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}\, dx$$

FOR $x < 0$ SUBSTITUTE $f(x) = -1$ AND for $x > 0$ LET $f(x) = +1$

$$A_N = \frac{1}{c}\int_{-c}^{0} -\sin\frac{n\pi x}{c}\, dx + \frac{1}{c}\int_{0}^{c} 1 \sin\frac{n\pi x}{c}\, dx$$

ALL WE HAVE DOWN IS JUST BREAK $f(x)$ UP INTO TWO CONTINUOUS PIECES. EVALUATING THE A_N'S

$$A_N = \frac{1}{n\pi}\left[\cos\left(\frac{n\pi x}{c}\right)\Big|_{-c}^{0}\right] - \frac{1}{n\pi}\left[\cos\frac{n\pi x}{c}\right]_{0}^{c}$$

for $c = 1$

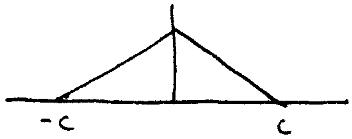
$$A_n = \frac{1}{n\pi}\left[1 - (-1)^n - (-1)^n + 1\right] = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

THIS IS 0 IF n = EVEN AND for AND odd n THE A_N BECOME $\frac{4}{n\pi}$. THUS $f(x)$ BECOMES

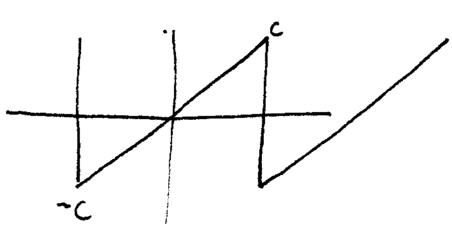
$$f(x) = \frac{4}{\pi}\sin\frac{\pi x}{\cdot} + \frac{4}{3\pi}\sin 3\pi + \frac{4}{5\pi}\sin 5x + \cdots$$

A PHYSICAL EXAMPLE of THIS FUNCTION IS A SWITCH TURNING VOLTAGE ON AND off.

TWO MORE USEFUL FUNCTIONS ARE THE SYMMETRICAL AND ANTI SYMMETRICAL SAWTOOTHS

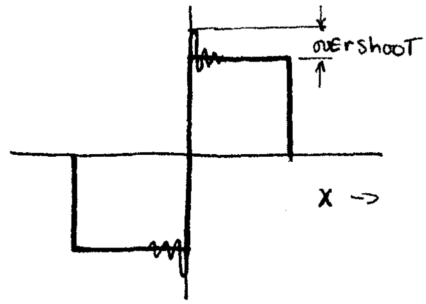


$$f(x) = \frac{c}{2} - \frac{4c}{\pi^2}\left[\cos\frac{\pi x}{c} + \frac{1}{9}\cos\frac{3\pi x}{c} + \cdots\right.$$



$$f(x) = \frac{2c}{\pi}\left[\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \cdots\right.$$

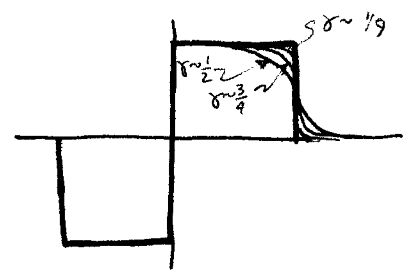
There has been a lot of mathematical discussion on the convergence of the fourier series. The question is after summing to a large N and then stopping how close to we come to f(x), i.e, how important is the rest of the series that you threw away. As an example the square wave function has a convergence but produces an overshoot which reaches some limiting value



There is a way around the overshoot difficulty and that is to attenuate the amplitude of the higher order terms. Thus their effect is less and the limiting behavior of the series is much smoother. We can put a factor γ into the fourier series and have

$$f(x) = \sum_{n=1}^{\infty} A_n \gamma^n \sin \frac{n\pi x}{c}$$

As $\gamma \to 1$ the function is approached in the limit. The series sum then looks like the following:



You can't always expect the mathematics to give you the right answer. You must understand the problem you are trying to solve. If the application of this fourier series involved some threshold detecting circuit, the overshoot phenomena could be quite troublesome. Using the γ-attenuation method would be required. On the other hand if you just wanted the area under square wave the overshoot would not be any trouble since a positive and negative overshoot cancel eachother out.

Fourier series have a lot of useful application one which is intereting but not often used it in summing series. For example if I had the series

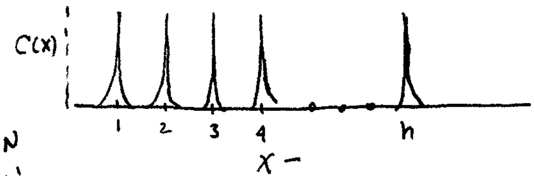
$$\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \cdots \frac{1}{1+N^2} = S$$

I CAN find S USING fourier ANALYSIS. NOW SUPPOSE THERE IS SOME FUNCTION $F(n)$ which reproduces The Above SERIES, i.e

$$\sum_n F(n) = F(0) + F(1) + F(2) + \cdots + F(n-1)$$

EACH INTEGER CAN BE EXPRESSED AS A DELTA FUNCTION SO LET ME DEFINE $C(x)$ AS

$$C(x) = \sum_{n=-\infty}^{+\infty} \delta(x-n)$$



If WE CONSIDER $F(n)$ TO BE A FUNCTION of x Then WE CAN retrieve The $F(n)$'s by

$$\int C(x)\, F(x)\, dx = \sum_n F(n)$$

WE WANT ONLY The n's >0 so The SUM ON $\delta(x-n)$ from $-\infty$ TO $+\infty$ MUST BE CUT IN HALF SO

$$C(x) = \frac{1}{2} \sum_{n=0}^{+\infty} \delta(x-n)$$

The FUNCTION $F(x)$ IS EVEN, i.e $F(x) = F(-x)$ SO The SUM S LOOKS LIKE

$$S = \frac{1}{2} F(0) + \frac{1}{2}\left(\ldots F(-2) + F(-1) + F(0) + F(1) + F(2) + \cdots \right)$$

$$= \frac{1}{2} F(0) + \frac{1}{2} \int_{-\infty}^{\infty} C(x)\, F(x)\, dx$$

The DELTA FUNCTION CAN BE EXPANDED AS A fourier SERIES between $-1/2$ And $+1/2$. SINCE $\delta(x) = \delta(-x)$ WE HAVE A COSINE SERIES

$$C(x) = \sum_k \delta(x-n) = \sum \tfrac{1}{2} b_0 + b_1 \cos\frac{\pi x}{c} + b_2 \cos\frac{2\pi x}{c} + \cdots$$

where The coefficients b_k's Are GIVEN AS

$$b_k = \frac{2}{c} \int_{-1/2}^{1/2} \delta(x) \cos\frac{kx\pi}{c}\, dx = 2$$

Therefore

$$C(x) = 1 + \sum_{k=1}^{\infty} 2\cos 2\pi k x$$

The SUMMATION formula becomes

$$S = \frac{1}{2} F(0) + \int_0^{\infty} F(x)\, dx + \sum_{k=1}^{\infty} \int_0^{\infty} F(x)\, 2\cos 2\pi k x\, dx$$

NOW YOU HAVE TO DO The INTEGRALS which MAY NOT BE AS EASY AS SOME OTHER TECHNIQUE. BUT SOMETIMES IT IS And when IT IS This APPROACH IS QUITE ACCURATE. AS $F(x)$ GETS SMOOTHER AND SMOOTHER The SERIES CONVERGES IN ONLY A few TERMS.

AS A problem FIND The SUM $S = 1 + e^{-a^2} + e^{-4a^2} + e^{-9a^2} + \cdots$

COMPLEX NOTATION OF FOURIER SERIES

I NOW WANT TO IMPROVE THE DEFINITION OF THE FOURIER SERIES BY WORKING IN COMPLEX NOTATION. THE FUNCTION $f(x)$ BECOMES

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{in\pi x}{c}}$$

IN THIS NOTATION IT IS EASIER TO WORK OUT THE MATHEMATICS. TO FIND THE COEFFICIENTS WE HAVE

$$\int_{-c}^{c} f(x)\, e^{-im\pi x/c}\, dx = \sum a_n \int e^{i(n-m)\pi x/c}\, dx$$

HERE ON THE RHS IS AN OSCILLATORY EXPONENTIAL WHICH WILL NOT CONTRIBUTE ANYTHING UNLESS $m=n$. WE GET

$$a_n = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{-im\pi x/c}\, dx$$

IF $f(x)$ IS A REAL FUNCTION THEN ITS COMPLEX CONJUGATE $f^*(x)$ IS GIVEN BY $f^*(x) = \sum a_n^* e^{-in\pi x/c}$. SINCE $F(x) = F^*(x)$ THE TWO SERIES MUST BE EQUAL THEREFORE THEIR COEFFICIENTS MUST BE EQUAL, I.E THE a_n'S MUST SATISFY $a_n = a_n^*$.

THE ENERGY THEOREM

ONE USEFUL THEOREM IS CALLED THE ENERGY THEOREM WHICH INVOLVES THE ABSOLUTE SQUARE OF $F(x)$, I.E.

$$E = \frac{1}{2c}\int_{-c}^{c} F(x)\, F^*(x)\, dx = \frac{1}{\cancel{2c}} \sum a_n \int e^{\frac{in\pi x}{c}} \sum_m a_m^* \int e^{-\frac{im\pi x}{c}} dx$$

NOW RECALL THAT $\int e^{\frac{im\pi x}{c}}\, e^{-\frac{im\pi x}{c}}\, dx = \delta_{nm} = 1$ for $m=n$ or 0 if $m \neq 0$

THUS WE HAVE

$$\frac{1}{2c}\int f(x)\, F^*(x)\, dx = \sum a_n a_n^*$$

or

$$E = \int_{-c}^{c} |f(x)|^2 \frac{dx}{2c} = \sum |a_n|^2 = 1$$

PHYSICALLY WHAT THIS MEANS IS IF WE HAVE A COMPLICATE WAVE WHICH CAN BE BROKEN UP INTO A LOT OF SINE WAVES OF DIFFERING AMPLITUDES, a_n, (THE INTENSITY OF EACH WAVE), THE TOTAL ENERGY CARRIED IN THE WAVE IS THE SUM OF THE SQUARE OF EACH COMPONENT INTENSITY. THE SUM OF ALL THE SQUARES MUST EQUAL 1, I.E. THE TOTAL ENERGY IN THE WAVE

You should be aware of your expanding power to solve problems. Recall we found for a square wave the series expansion was

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \cdots \right.$$

The average square is equal to 1 so

$$|f(x)|^2 = 1 = \frac{16}{\pi^2}\left[\frac{1}{2}\cdot\frac{1}{1^2} + \frac{1}{2}\frac{1}{3^2} + \frac{1}{2}\frac{1}{5^2} + \cdots \right] = \frac{8}{\pi^2}\sum_{n\,odd}\frac{1}{n^2}$$

Thus we have found

$$\sum_{n\,odd}\frac{1}{n^2} = \frac{\pi^2}{8}$$

There is a generalization to the energy theorem which is worth know and that is given $f(x)$ and $g(x)$ the energy is given by

$$E = \frac{1}{2c}\int_{-c}^{c} f(x)\, g^*(x)\, dx = \sum a_n b_n^*$$

where

$$f(x) = \sum a_n e^{in\pi x/c}, \quad g(x) = \sum b_n e^{in\pi x/c}$$

Returning to the question of convergence again we can do a better job now. Suppose that $f(x)$ is series expanded over the interval of $-\pi$ to π then

$$f(x) = \sum a_n e^{inx}$$

Lets try to find in terms of $f(x)$ what $f_N(x)$ is where N is some finite number where the summing stops. $F_N(x)$ is defined as

$$F_N(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

The coefficients are given by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\, dt$$

so that

$$F_N(x) = \sum_{-N}^{N} e^{inx} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\, dt$$

It appears that things are getting worse for us but if we pull the integral out front then we have the geometric series $\sum_{-N}^{N} e^{in(x-t)}$ which can be written as

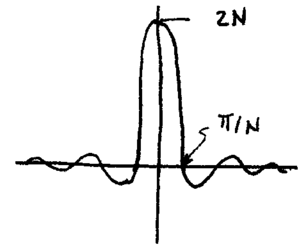
$$\sum e^{in(x-t)} = \frac{e^{-i(N+1/2)(x-t)} - e^{i(N+1/2)(x-t)}}{e^{-1/2(x-t)} - e^{1/2(x-t)}}$$

And simplifying we have

$$F_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+1/2)(x-t)}{\sin 1/2\,(x-t)}\, f(t)\, dt$$

THIS SAYS TO TAKE $F(t)$ AND ADD TOGETHER WITH THE WEIGHTING FACTOR OF $\dfrac{\sin(N+1/2)(x-t)}{\sin 1/2(x-t)}$. IN THE CASE OF $F(t)$ BEING A STEP FUNCTION WE HAVE A FORMULA FOR FINDING THE OVERSHOOT

$$F_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+1/2)(x-t)}{\sin 1/2(x-t)}\, dt$$



FOURIER TRANSFORMS

So far we have talked about functions which were represented between $-c$ and c. I am not interested in letting the range expend to ∞ and ask if there is still a representation of the function. The answer is indeed yes but in order to make the representation more exact I'll need a lot of n's to keep the exponential power from going to zero, i.e. $e^{+in\pi x/c}$ as $c \to \infty$ n must $\to \infty$. If we replace $\frac{n\pi}{c}$ by ω a new number then as c increases ω gets closer. In this process the coefficients of a_n are converted to a function of ω, $\varphi(\omega)$, while the summation over n becomes an integral over ω, i.e.

$$\sum_n \longrightarrow \int d\omega \left(\frac{c}{\pi}\right)$$

Thus we have for $f(x)$,

$$f(x) = \int_{-\infty}^{\infty} \frac{\varphi(\omega)}{2\pi}\, e^{+i\omega x}\, d\omega$$

where $2c\,a_n \equiv \varphi(\omega)$

This representation of $f(x)$ involves the superposition of an infinite number of oscillatory waves. If we want to recover $\varphi(\omega)$ we need to evaluate

$$\varphi(\omega) = 2c\,a_n = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx$$

This is the inverse fourier transform.

In the new representation the energy theorem becomes

$$\int_{-\infty}^{\infty} f(x) f^*(x)\, dx = \sum 2c(a_n a_n^*) = \int a_n a_n^*\, d\omega \left(\frac{2c^2}{\pi}\right) = \int_{-\infty}^{\infty} \varphi(\omega)\varphi^*(\omega)\frac{d\omega}{2\pi}$$

SOLVING DIFFERENTIAL EQUATIONS BY FOURIER TRANSFORM TECHNIQUE

SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS IS A COMMON OCCURRENCE IN PHYSICS AND ENGINEERING. ANY TECHNIQUE WHICH MAKES THE TAKE EASIER IS APPRECIATED. THE NORMAL EQUATION INVOLVES DERIVATIVES OF SOME VARIABLE y AND CAN BE EXPRESSED AS

$$\Sigma\, a_n \frac{d^n y}{dx^n} = g(x)$$

THE RHS IS REFERRED TO THE INHOMOGENEOUS PART OF DIFFERENTIAL EQUATION. THE HOMOGENEOUS EQUATION IS WHEN $G(x) = 0$. AS AN EXAMPLE OF THE DIFFERENTIAL EQUATION IS

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \qquad \text{for } x<0$$
$$= e^{-2x} \qquad \text{for } x>0$$

A LINEAR DIFFERENTIAL EQUATION HAS THE PROPERTY OF A SOLUTION WHICH IS THE SUM OF PIECES. FOR EXAMPLE $L(Y) = L(Y_1 + Y_2) = L(Y_1) + L(Y_2)$. THIS PROPERTY EXISTS WHEN THE COEFFICIENTS DO NOT DEPEND ON X.

THERE ARE HUNDREDS OF WAYS TO SOLVE THESE EQUATIONS WHICH CAN BORE YOU DEATH. BUT SINCE THERE IS ONLY ONE ANSWER THE EASIEST WAY TO THAT ANSWER IS MOST ATTRACTIVE. FOURIER TRANSFORMS ARE VERY USEFUL IN SOLVING THESE KINDS OF PROBLEMS. THE PRINCIPLE UPON WHICH THIS TECHNIQUE IS BASED IS THE IDEA OF REPRESENTING A FUNCTION $f(x)$ BY A FOURIER TRANSFORM $\varphi(\omega)$ SUCH THAT

$$f(x) = \int_{-\infty}^{\infty} e^{i\omega x}\, \varphi(\omega)\, \frac{d\omega}{2\pi}$$

DIFFERENTIATING $f(x)$ WE FIND THAT

$$f'(x) = \int_{-\infty}^{-\infty} i\omega\, e^{i\omega x}\, \varphi(\omega)\, \frac{d\omega}{2\pi} = i\omega\, f(x)$$

IN OTHERWORDS DIFFERENTIATING CAN BE SIMPLIED TO MULTIPLICATION BY $i\omega$. THEN THE PROBLEM IS REDUCED TO ONE OF ALGEBRAIC MANIPULATION.

The algebraic manipulation can go both ways between the function, its transform, and back again. Consider the following two transforms

$$G(\omega) = \int g(x)\, e^{-i\omega x}\, dx$$
$$Y(\omega) = \int y(x)\, e^{-i\omega x}\, dx$$

we know that

$$i\omega\, Y(x) = \int_{-\infty}^{\infty} \left(\frac{dy}{dx}\right) e^{-i\omega x}\, dx$$

but also

$$-i\, Y'(\omega) = \int x\, y(x)\, e^{-i\omega x}\, dx$$

Returning to the differential equation, if we multiply both sides by $e^{-i\omega x}\, dx$ and integrate we get

$$\sum a_n (i\omega)^n\, Y(\omega) = G(\omega)$$

we can define $P(i\omega)$ as $\sum a_n (i\omega)^n$ which is a polynomial in $i\omega$. We can solve for $Y(\omega)$ then as

$$Y(\omega) = \frac{G(\omega)}{P(i\omega)}$$

we have the fourier transform of the answer so we have to transform back again

$$y(x) = \int_{-\infty}^{\infty} \frac{G(\omega)}{P(i\omega)}\, e^{i\omega x}\, \frac{d\omega}{2\pi}$$

A set of tables for working out these fourier transforms is very useful. What we formally have to do is substitute in for $G(\omega)$ its transform;

$$y(x) = \int \frac{e^{i\omega x}}{P(i\omega)} \int g(x')\, e^{-i\omega x'}\, dx'\, \frac{d\omega}{2\pi}$$

If we reverse the order of integration

$$y(x) = \int g(x')\, dx' \int \frac{e^{i\omega(x-x')}}{P(i\omega)}\, \frac{d\omega}{2\pi}$$

now define

$$R(x-x') = \int \frac{e^{i\omega(x-x')}}{P(i\omega)}\, \frac{d\omega}{2\pi}$$

To evaluate $R(x-x')$ it is necessary to calculate the integral of a polynomial. The easiest way to do that is by residue theory and contour integration. Since $P(i\omega)$ is a polynomial it can be written as

$$P(i\omega) = a_n (i\omega - \alpha_1)(i\omega - \alpha_2) \cdots (i\omega - \alpha_n)$$

WE WILL FIRST ASSUME NO 2 ROOTS ARE THE SAME. THUS EVERY TIME WE HAVE A POLE AT $\alpha_1, \alpha_2, \ldots \alpha_n$, WE NEED TO DETERMINE THE RESIDUE. THE RESIDUE IS THE POLYNOMIAL MISSING THE POLE AT WHICH THE EVALUATION IS MADE WITH THE REMAINING POLYNOMIAL EVALUATED AT $i\omega = \alpha_r$. AS IT TURNS OUT

$$\left. \frac{i\omega - \alpha_r}{P(i\omega)} \right|_{i\omega \to \alpha_r} = \frac{1}{P'(\alpha_r)} = \text{residue}$$

SO THAT THE SOLUTION TO THE DIFFERENTIAL EQUATION IS

$$Y(x) = \sum_r \frac{1}{P'(\alpha_r)} \int_{?}^{x} e^{\alpha_r (x - x')} g(x')\, dx$$

AS MENTIONED EARLIER WE CAN WRITE THE SOLUTION AS A SUM OF PIECES; ONE PIECE CAN ALWAYS BE THE HOMOGENEOUS SOLUTION, I.E. $L(Y_1) = 0$, $L(Y_2) = G(x)$ THEN $L(Y_1 + Y_2) = G(x)$. THE SOLUTION BECOMES

$$Y(x) = \sum \frac{1}{P'(\alpha_r)} e^{\alpha_r x} \int_c^x e^{-\alpha_r x'} g(x')\, dx' + \sum A_r e^{\alpha_r x}$$

WHERE $e^{\alpha_r x}$ IS THE HOMOGENEOUS SOLUTION.

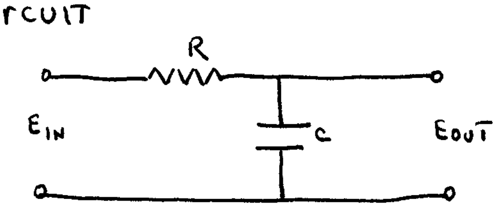
THERE IS ANOTHER WAY TO WRITE OUT THE DIFFERENTIAL EQUATION WHICH IS OFTENTIMES USEFUL. IF WE USE THE NOTATION THAT $DY = dY/dx$, THEN THE DIFFERENTIAL EQUATION BECOMES

$$\sum a_n D^n Y(x) = g(x)$$

OR

$$P(D)\, Y(x) = g(x)$$

LET'S WORK OUT A SIMPLE EXAMPLE OF A LINEAR DIFFERENTIAL EQUATION OF THE 1ST DEGREE. CONSIDER THE FOLLOWING ELECTRICAL CIRCUIT



WHERE E_{IN} IS THE INPUT VOLTAGE AND E_{OUT} IS THE OUTPUT VOLTAGE. THE TIME HISTORY OF CIRCUIT IS GIVEN BY

$$RC \frac{d E_{OUT}}{dt} + E_{OUT} = E_{IN}(t)$$

LETTING $D = d/dt$ WE CAN WRITE

$$(RC\,D + 1)\, E_{OUT} = E_{IN}$$

HERE $P(D) = P(\alpha_r) = RC\,D + 1$. SOLVING for α WE HAVE

$$\alpha = -\frac{1}{RC}$$

Thus There is ONLY 1 root And $P'(\alpha) = RC$. Therefore The SOLUTION IS

$$E_{OUT} = \frac{1}{RC} e^{-t/RC} \int_c^t e^{t'/RC} E_{IN}(t')\, dt' + A e^{-t/RC}$$

To go on we need The INITIAL conditions of The problem. Suppose $E_{OUT} = E_{IN} = 0$ At $t = 0$. Then

$$E_{OUT} = \frac{1}{RC} e^{-t/RC} \int_{-\infty}^{t} e^{t'/RC} E_{IN}(t')\, dt'$$

As A CHECK

$$RC \frac{d E_{OUT}}{dt} = -\frac{1}{RC} e^{-t/RC} \int_{-\infty}^{t} e^{t'/RC} E_{IN}(t')\, dt + e^{-t/RC} e^{t/RC} E_{IN}(t)$$

ANOTHER EXAMPLE IS TO EVALUATE $S(m) = \int_0^\infty \frac{\sin mt}{1+t^2}\, dt$ If we differentiate TWICE WE HAVE

$$S''(m) = \int -t^2 \frac{\sin mt}{1+t^2}\, dt$$

Adding $S(m)$ TO both Sides

$$-S''(m) + S(m) = \int_0^\infty \sin mt\, dt = 1/m$$

The Problem IS Then

$$S''(m) - S(m) = -1/m$$

or ALTERNATELY

$$Y''(x) - Y(x) = -\frac{1}{x} = G(x)$$

IN D NOTATION WE have

$$D^2 - 1 = -1/x$$

where $P(D) = D^2 - 1$. FINDING The roots WE GET

$$\alpha_1 = +1 \qquad \alpha_2 = -1$$

SO THAT $P'(\alpha_1) = 2D = 2$ AND $P'(\alpha_2) = -2$

$$Y(x) = \frac{e^x}{2} \int_c^x \left(-\frac{1}{x'}\right) e^{-x'}\, dx' - \frac{1}{2} e^{-x} \int^\infty \frac{e^{+x'}}{x'}\, dx'$$

The integrals cannot be evaluated directly but can be expressed as error integral E_i previously discussed. Thus $S(m)$ can be written as

$$S(m) = \frac{1}{2}\left[e^{m} E_i(-m) - e^{-m} E_i(m) \right] + A e^{-m} + B e^{m}$$

If $-S(m) = S(-m)$ by symmetry we deduce $B = -A$. Also we know as $m \to \infty$ the oscillations must damp out so $B = A = 0$.

Now for the case of double roots we must add another term to the solution. That is if $D^2 = 1$ has the double root $\alpha_0 = 1$, the solution is given by

$$Y(x) = \int^{x} \frac{e^{\alpha_0 (x-x')}}{P''(x)} (x-x')\, g(x')\, dx' + A e^{\alpha_0 x} + B x e^{\alpha_0 x}$$

The polynomial $P(D)$ will not always have real roots. In such a case as $D^2 = -1$ where $D = \pm i$ put $\alpha_n = \pm i$ and go ahead. You will get sines and cosines. If the answer has real roots they will appear at the end. Don't get nervous with exponentials and complex coefficients.

In order to expand our knowledge we will talk briefly about differential equations with variable coefficients. If $\varphi(\omega)$ is the fourier transform of $F(x)$ then the following transformations hold

$$i\omega\, \varphi(\omega) \longleftrightarrow F'(x)$$
$$(i\omega)^n \varphi(\omega) \longleftrightarrow f^n(x)$$
$$i\, \varphi'(\omega) \longleftrightarrow x\, F(x)$$
$$-\varphi''(\omega) \longleftrightarrow x^2 F(x)$$
$$i\omega\, \cdot \longleftrightarrow \frac{d}{dx}$$
$$i\frac{d}{d\omega} \cdot \longleftrightarrow x$$

Thus we see that multiplying by $i\omega$ is the equivalent to differentiating in x space. Conseruely differentiating in ω space is equivalent to multiplying by x in x-space.

We can use the technique of differentiating in ω-space to solve some special non-linear differential equation. For example, suppose we want to solve

$$\frac{d^2Y}{dX^2} + XY = 0$$

Because this is nonlinear with X to first power we can transform and get

$$(i\omega)^2\, Y(\omega) + i\,\frac{d\,Y(\omega)}{d\omega} = 0$$

We have reduced the order of differential equation by one so it is easier to solve. We can solve for $Y(\omega)$ and get

$$\frac{dY(\omega)}{Y(\omega)} = -i\omega^2 d\omega \quad \rightarrow \quad Y(\omega) = A e^{-\frac{i}{3}\omega^3}$$

Now to find $Y(X)$ we must transform back, i.e.,

$$Y(X) = A \int e^{-i\omega^3/3}\, e^{i\omega x}\, \frac{d\omega}{2\pi}$$

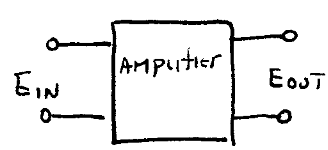
This integral is related to the Bessel function of 1/3 order, sometimes called the Aires integral.

As a problem try to solve

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xY = 0$$

I should point out that divergent solutions do not have Fourier transforms. For example Ae^{x^2} has not F.T.

I now want to discuss a class of black boxes called amplifiers. The amplifier has the property that E_{OUT} is related to $E_{IN}(t)$ by some quantity g, sometimes referred to as the gain of the device, i.e.

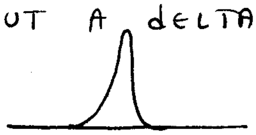


$$E_{OUT} = g\,E_{IN}$$

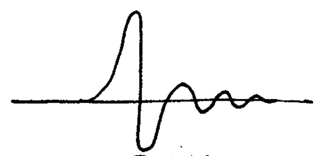
g has a number of properties one of which is often linearity. That is if $f_1(t)$ is put in then F_1 is out. And if $f_2(t)$ is in then F_2 is out so that F_1+F_2 in gives F_1+F_2 out. Simply stated if you double the input you double the output. This is a linear amplifier

The amplifier has the property of being time invariant. If $f(t)$ is in at t then $F(t)$ is out. ~~The~~ The sample signal is put in at time $t+a$, the same output is obtained.

The amplifier can be analyzed by investigating its behavior to a special function, the delta function, which permits deducing the response for other input functions. If we put a delta function the output will look like



$\delta(t)$ IN



$R(t)$ OUT

Some properties between $\delta(t)$ and $R(t)$ are the following

$$\delta(t+t_0) \rightarrow R(t+t_0)$$
$$b\,\delta(t-t_0) \longrightarrow b\,R(t-t_0)$$
$$\delta(t-t_1) + \delta(t-t_2) \longrightarrow R(t-t_1) + R(t-t_2)$$
$$f(t_1)\,\delta(t-t_1) \longrightarrow f(t_1)\,R(t-t_1)$$

The idea we are developing is by putting a whole bunch of delta functions together each at different times and of different amplitudes we can construct any wave packet you want. Thus we can write

$$F_{out}(t) = \int R(t-t')\, f_{in}(t')\, dt'$$

Once the response to an impulse is established it is possible to solve all subsequent equations given the system transfer function.

MORE ON THE AMPLIFIER AND THE CAUSAL IMPLICATIONS ON THE TRANSFER FUNCTION.

LAST TIME WE STARTED TO DISCUSS AN AMPLIFIER BY USING FOURIER TRANSFORM THEORY. I WANT TO CONTINUE THAT SUBJECT.

BASICALLY AN AMPLIFIER INVOLVES TWO FUNCTIONS WHICH ARE RELATED BY SOME OTHER FUNCTION, SOMETIMES REFERRED TO AS A GREEN'S FUNCTION, $R(t)$. THIS IS THE RESPONSE OF THE AMPLIFIER TO AN IMPULSE INPUT. A TABLE OF INPUT AND OUTPUT FUNCTIONS IS USEFUL TO SUMMARIZED

IN FUNCTION	OUT FUNCTION
$\delta(t)$, DELTA	$R(t)$
$\delta(t-a)$	$R(t-a)$
$\int f(t')\,\delta(t-t')\,dt'$	$\int f(t')\,R(t-t')\,dt'$

THESE RELATIONSHIPS IMPLY IF I INPUT A SIGNAL WITH CONSTANT FREQUENCY, ω, I.E $E_{IN} = e^{i\omega t}$; THEN I WILL GET OUT THE SAME FREQUENCY BUT AMPLIFIED AND PERHAPS PHASE SHIFTED, I.E $E_{OUT} = A(\omega)\,e^{i\omega t}$. THE FUNCTION $A(\omega)$ IS COMPLEX SUCH THAT ITS MAGNITUDE IS CALLED THE AMPLIFICATION WHILE THE IMAGINARY PART YIELDS A PHASE SHIFT. $A(\omega) = |A(\omega)|\,e^{i\varphi(\omega)}$ $A(\omega)$ IS OFTEN CALLED THE TRANSFER FUNCTION OF THE AMPLIFIER.

IF THERE ARE ALOT OF FREQUENCIES AT THE INPUT THEN $E_{IN} = \varphi(\omega)\,e^{i\omega t}$ AND THE OUTPUT WILL DEPEND ON THE AMPLITUDE OF EACH OF THE VARIOUS COMPONENTS. WE MUST INTEGRATE OVER ALL FREQUENCIES TO GET THE TOTAL OUTPUT SIGNAL

$$\int \varphi(\omega)\,e^{i\omega t}\,\frac{d\omega}{2\pi} \longrightarrow \int \varphi(\omega)\,A(\omega)\,e^{i\omega t}\,\frac{d\omega}{2\pi}$$

WE CAN STUDY THIS BEHAVIOR IN MORE DETAIL BY USING THE FACT THAT

$$f(t) = \int f(t')\,R(t-t')\,dt'$$

NOW LET $f(t) = e^{i\omega t}$ SUCH THAT THE TRANSFER IS $\int e^{i\omega t'}R(t-t')\,dt'$ BY REDEFINING THE TIME BASE TO BE $\tau = t-t'$ WE CAN WRITE THE INTEGRAL AS

$$e^{i\omega t}\int_0^\infty e^{-i\omega \tau}\,R(\tau)\,d\tau$$

Thus if $E_{IN} = e^{i\omega t}$ And $E_{OUT} = A(\omega)\, e^{i\omega t}$ we have found That

$$A(\omega) = \int e^{-i\omega \tau} R(\tau)\, d\tau$$

The Green function $R(\tau)$ can be found by the inverse Transform

$$R(\tau) = \int e^{i\omega \tau} A(\omega) \frac{d\omega}{2\pi}$$

In the case many frequencies are present we have that

$$E_{IN} = f(t) = \int \varphi(\omega)\, e^{i\omega t} \frac{d\omega}{2\pi}$$

And

$$\varphi(\omega) = \int f(t)\, e^{-i\omega t}\, dt$$

To find E_{OUT},

$$E_{OUT} = \int \varphi(\omega) A(\omega)\, e^{i\omega t} \frac{d\omega}{2\pi} = \int \left[\int f(t')\, e^{-i\omega t'} dt' \right] A(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$$

$$= \int f(t')\, dt'\, R(t-t')$$

In order to proceed we need to take the Fourier Transform of the Product of two functions. This is a useful concept so let me generalize by calling the two functions $f(t)$ and $g(t)$ both have F.T.'s: $f(t) \longrightarrow F(\omega)$ And $g(t) \longrightarrow G(\omega)$

Then we want

$$\int f(t')\, g(t-t')\, dt' \longrightarrow F(\omega)\, G(\omega)$$

The product $f(t')\, g(t-t')$ is called a convolution. It is useful in many cases to fish around in the ω-space which have easy F.T.'s since the integral reduces to multiplication of two functions.

In summary we have established that the characteristics of an amplifier can be obtained by knowing its response to an impulse or to a sine wave of definite frequency.

It is interesting to study the case when $R(\tau)=0$ for $\tau<0$. This is a statement that no response will occur until an input signal is applied. This then as a statement of causality and implies that $A(\omega)$ has certain characteristics. We require then

$$\int_{-\infty}^{0} R(t)\, e^{-i\omega'\tau}\, d\tau = 0$$

Substituting for $R(\tau)$ this equality becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{0} A(\omega)\, e^{i\omega\tau}\, e^{-i\omega'\tau}\, d\tau\, \frac{d\omega}{2\pi}$$

The integral over τ can be evaluated by putting in a converging factor $e^{\epsilon\tau}$ (note $\tau<0$) and taking the limit as $\epsilon \to 0$. Thus we have to evaluate

$$\int_{-\infty}^{0} e^{i(\omega-\omega')\tau}\, e^{\epsilon\tau}\, d\tau$$

The integral can be evaluated as

$$\lim_{\epsilon\to 0} \frac{1}{i(\omega-\omega')+\epsilon}$$

Putting this back into the integral

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{A(\omega')\, d\omega'/2\pi}{i(\omega'-\omega)+\epsilon} = 0$$

This can be written as

$$\int \frac{A(\omega')\, d\omega'/2\pi}{\omega-\omega'-i\epsilon} = 0$$

And the integral is in the form of a convolution between $A(\omega)$ and $\frac{1}{\omega-\omega'}$. Its F.T in time space is just

$$\mathbb{1}(-t)\, R(t) = 0$$

where $\mathbb{1}(-T)$ is the unit step function which is 0 for $t<0$.

The condition on $A(\omega)$ is that it have no singularities below the real axis. Thus for $R(\tau)$ to be causal $A(\omega)$ is not arbitrary but rather satisfies the criteria of no poles below the real axis.

The notion of a complex frequency, $\omega = \omega_R + i\,\omega_I$ is very useful here and to pursue it is worthwhile. ω represents a wave whose amplitude is changing exponentially with time. If ω_I is > 0, the amplitude is decaying with time.

Now a complex function can have a singularity called a pole. This is a point at which the function is asymptotically infinite. A pole in a physical function represents a resonance, which is a frequency at which the amplitude of oscillation becomes infinite for a driving force of finite amplitude.

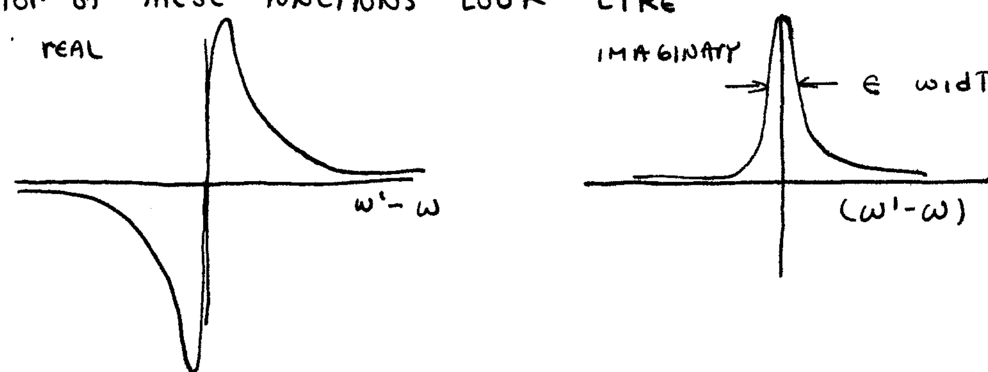
Any physical system has a "memory" which lasts a certain while; it is impossible that a physical system should have precognizance of events to come. This is just a statement of the principle of causality. The only way a physical system can achieve infinite amplitude is the result of its memory of an infinite driving force at some earlier time. Since a pole represents infinite amplitude of oscillation for a finite driving force; hence it must arise from a force that has exponentially decreased from infinite amplitude at $t = -\infty$. This implies the driving force has a complex frequency with positive ω_I. Thus the poles of a real system must lie in the upper half plane of the complex frequency space, corresponding to decaying amplitudes. Said another way the oscillations of all real physical systems decay naturally with time, the resonance frequencies all have positive ω_I and lie in the upper half plane.

Ref. Optical Physics, S.G. Lipson and H. Lipson, Cambridge U. Press.

Let's now deduce some properties of A by investigating the limiting behavior of the integrand. Let's write the factor $\frac{1}{\omega'-\omega-i\epsilon}$ as the sum of an imaginary and real part:

$$\frac{1}{\omega'-\omega-i\epsilon} = \underset{\text{(real)}}{\frac{\omega'-\omega}{(\omega'-\omega)^2+c^2}} + \underset{\text{(imaginary)}}{\frac{i\,\epsilon}{(\omega'-\omega)^2+\epsilon^2}}$$

The behavior of these functions look like



The real part in the limit goes as the principal value, i.e. PV $\frac{1}{\omega'-\omega}$ since there are equal distances on either side of the singularity. The imaginary part behaves as a delta function as $\epsilon \to 0$ while the area goes to π. Thus we have as the limiting result

$$\frac{1}{\omega'-\omega-i\epsilon} \to PV\frac{1}{\omega'-\omega} + i\pi\,\delta(\omega'-\omega)$$

Thus if I understand the principal value idea I have a result that looks like

$$0 = \int_{-\infty}^{\infty} A(\omega')\, PV\frac{1}{\omega'-\omega}\,\frac{d\omega'}{2\pi} + i\pi\int A(\omega')\,\delta(\omega'-\omega)\,\frac{d\omega'}{2\pi}$$

The imaginary part is easy to integrate; it is just $\frac{i\,A(\omega)}{2}$. The real part needs to be integrated around the singularity as

$$\int_{-\infty}^{\omega-\delta} + \int_{\omega+\delta}^{\infty} A(\omega')\, PV\frac{1}{\omega'-\omega}\,\frac{d\omega'}{2\pi}$$

This is a little difficult to integrate so for the moment let me write the result as

$$\int_{-\infty}^{\infty} A(\omega')\, PV\frac{1}{\omega'-\omega}\,\frac{d\omega'}{4\pi} = -i\,A(\omega)$$

Now $A(\omega)$ is complex so we can write it as the sum of a real and imaginary part, i.e.

$$A(\omega) = A_R(\omega) + i\,A_I(\omega)$$

And from the previous result we have the condition that for the real part,

$$\int_{-\infty}^{\infty} A_R(\omega')\,\frac{PV}{\omega'-\omega}\,\frac{d\omega'}{\pi} = A_I(\omega)$$

Thus given the real or imaginary part the other can be solved for. That is, for the real part,

$$-\int_{-\infty}^{\infty} A_I(\omega')\;PV\,\frac{1}{\omega'-\omega}\cdot\frac{d\omega'}{\pi} = A_R(\omega)$$

An example of these relations which have been called the dispersion relations the index of refraction of light is used. In this case the imaginary function corresponds to the adsorption through a media while the real part is the refraction index, n. From these relations we learn that n varies as a function of frequency. This phenomena is referred to as chromatic aberration in optics.

I'll give one more example of the use of Fourier series before I leave the subject. I'll work out an example of heat flow. I will solve the one dimensional heat flow equation

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t}$$

subject to the initial condition that at $t=0$ the heat distribution is given by $f(x)$, i.e. $T(x,0) = f(x)$. Now fourier transforming $T(x,t)$ we have

$$\tau(x,t) = \int e^{ikx}\,T(x,t)\,dx$$

and

$$\tau(x,0) = \varphi(k)$$

The differential equation becomes

$$-k^2\,\tau(k,t) = \frac{d\tau}{dt}$$

THIS EQUATION IS NOW EASILY SOLVED AS

$$\tau(k,t) = e^{-k^2 t} A(k)$$

WHERE $\tau(k,0) = A(k) = \varphi(k)$. THEN $\tau(k,t) = \varphi(k)\, e^{-k^2 t}$
SINCE $e^{-k^2 t}$ IS LIKE A TRANSFER FUNCTION WE CAN INVERT IT TO FIND

$$T(k,t) = \int e^{-kxi} \tau(k,t) \frac{dk}{2\pi}$$
$$= \int e^{-kxi} e^{-k^2 t} \varphi(k) \frac{dk}{2\pi}$$

SINCE $\varphi(k) = \int e^{ikx'} f(x')\, dx'$ WE CAN REWRITE THE EXPONENTIAL AS

$$e^{-t\left[k + i\frac{(x-x')}{2t}\right]^2} e^{-\frac{(x-x')^2}{4t}}$$

AND FINALLY

$$T(x,t) = \int \sqrt{\frac{1}{4\pi t}}\; e^{-\frac{(x-x')^2}{4t}} f(x')\, dx'$$

THIS IS A CONVOLUATION INTEGRAL IN X BY A GAUSSIAN DISTRIBUTION.

PART TWO

I WILL NOW CHANGE THE SUBJECT AND START TO DISCUSS A SERIES OF NEW SUBJECTS. WHAT I HAD IN MIND WAS TO GO THROUGH THE FOLLOWING LIST, PASSING THROUGH SOME MORE RADILY THAN OTHERS DEPENDING UPON THE INTEREST:

- DIFFERENTIAL EQUATIONS
- PARTIAL DIFFERENTIAL EQUATIONS
- CALCULUS OF VARIATIONS
- INTEGRAL EQUATIONS
- MATRICES WITH APPLICATION TO VIBRATION THEORY
- EIGENVALUES OF LINEAR DIFF. EQ'S AND INTEGRAL EQ'S.
- PERTURBATION PROBLEMS IN LINEAR SYSTEMS.
- PROBABILITY AND STATISTICS.

DIFFERENTIAL EQUATIONS

A DIFFERENTIAL EQUATION IS A RELATIONSHIP BETWEEN TWO VARIABLES SAY X AND Y INVOLVING DERIVATIVES OF ONE WITH RESPECT TO THE OTHER, I.E AN EQUATION IN $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}$. IN GENERAL THE LOWER THE ORDER OF THE EQUATION, I.E, THE n OF THE DERIVATIVE, THE EASIER THE EQUATION IS TO SOLVE. USUALLY IN ATTACKING A DIFF. EQ. THE 1ST STEP IS TO REDUCE THE ORDER BY ONE. THERE IS ONE CASE, THE LINEAR DIFF. EQ WITH CONSTANT COEFFICIENTS, WHERE THIS COMPLICATES THE SOLUTION.

I WILL CONSIDER THE PROBLEM SOLVED WHEN THE SOLUTION HAS BEEN REDUCED TO AN INTEGRAL FORM. IN MOST CASES THE DIFF. EQ CANNOT BE EXACTLY SOLVED. IT'S ONLY IN THE TEXTBOOKS THAT YOU FIND SOLVABLE PROBLEMS. I'LL FIRST FOLLOW TRADITION BY DISCUSSING SOLUABLE PROBLEMS. YOU WILL SEE THAT SOLVING DIFF. EQ. REQUIRES A LOT OF FOOLING AROUND TRYING TO FIND THE EASIEST FORM FOR SOLVING THE EQUATION.

FIRST ORDER DIFFERENTIAL EQUATIONS

FIRST ORDER DIFF. EQ'S. ARE OF THE FORM $f(x, y, \frac{dy}{dx}) = 0$. OR MORE COMMONLY EXPRESSED AS $\frac{dy}{dx} = F(x,y)$. THE MOST GENERAL FIRST ORDER DIFF. EQ.'S ARE NOT SOLUABLE. THE EQUATION IS SOLUABLE IF THE ANSWER IS REDUCED TO QUADRATURE. OFTEN TIMES THE BEST APPROACH IS TO MAKE A TABLE OF x, y, AND dy/dx AND COMPUTE THE SLOPE FOR VARIOUS INTERESTING VALUES OF x AND y. AFTER YOU FIND HOW FAST y CHANGES WITH x, YOU CAN RECOMPUTE.

THE DIFF EQ DOES NOT COMPLETELY DEFINE THE FUNCTIONAL SOLUTION AND IN GENERAL YOU NEED SOME ARBITRARY CONSTANT OF INTEGRATION. AS THE ORDER OF EQUATION INCREASES, YOU WILL NEED A CORRESPONDING NUMBER OF SUCH CONSTANTS.

SOLVING DIFFERENTIAL EQUATIONS BY STUDYING THE CHARACTER OF THE SLOPE FOR VARIOUS VALUES OF x IS OFTEN QUITE INFORMATIVE. THIS IDEA HAS BEEN EXPANDED AND IS CALLED PASCALS DIAGRAM. HERE FOR EACH VALUE OF y AND x YOU DRAW A LITTLE SLOPE OF UNIT LENGTH. AFTER DRAWING A LOT OF THESE LITTLE SLOPES YOU CAN JOIN THEM UP AND GET A FEEL FOR THE SOLUTION AND ITS BEHAVIOR IN VARIOUS ~~REGU~~ REGIONS. THE LINES JOINING THE VARIOUS REGIONS ARE CRITICAL SOLUTIONS.

NOW WE'LL INVESTIGATE SOME METHODS OF SOLVING EXACTLY THOSE EQUATIONS WHICH CAN BE SOLVED EXACTLY.

CASE 1: NO Y. $\frac{dy}{dx} = F(x)$ THIS IS INTEGRATED DIRECTLY

CASE 2: NO X $\frac{dy}{dx} = F(y)$ INVERT AND INTEGRATE AS 1 $\frac{dx}{dy} = \frac{1}{F(y)}$

THEN $x = \int \frac{dy}{F(y)}$

CASE 3: $f(x,y) = -\frac{M(x,y)}{N(x,y)}$ WITH SPECIAL CONDITION $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

THIS IS NOT A VERY USEFUL CASE SINCE NOT MANY EQUATIONS HAVE THIS PROPERTY. THE DIFF. EQ IS SOLUABLE SINCE $dy/dx = -\frac{M}{N} \rightarrow M dx + N dy = 0$ AND IF $\varphi(x,y)$ SATISFIES

$$d\,\varphi(x,y) = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy$$

WHERE $M = \frac{\partial \varphi}{\partial x}$ AND $N = \frac{\partial \varphi}{\partial y}$

THE SOLUTION TO CASE 3 IS JUST $\varphi(x,y) = \int M\,dx + K(y)$

AS AN EXAMPLE

$$\frac{dy}{dx} = -\frac{y+2x}{1+x} \longrightarrow (y+2x)\,dx + (1-x)\,dy = 0$$

INTEGRATING $d(yx + x^2 + y) = 0$ WE GET

$$yx + x^2 + y = C \quad \text{or} \quad y = \frac{C - x^2}{1+x}$$

CASE 4: $\dfrac{dy}{dx} = \dfrac{g(y)}{f(x)} \longrightarrow \dfrac{dy}{g(y)} = \dfrac{dx}{f(x)}$

IN SOLVING NOT ONLY FIRST ORDER BUT ALSO HIGHER EQUATIONS IT IS SOMETIMES USEFUL TO CHANGE THE SCALE OF THE EQUATION BY CHANGING VARIABLES. FOR EXAMPLE CONSIDER

$$\frac{dy}{dx} = \frac{y^2}{x^2} + 1$$

IF x IS CHANGED TO cx AND y IS CHANGED BY A LIKE AMOUNT cy THE SCALE IS PRESERVED. SUCH AN EQUATION IS HOMOGENEOUS OR INVARIANT TO A SCALE & TRANSFORMATION. IN SUCH CASES THE SUBSTITUTION $y/x = v$ CAN BE MADE. THE SCALE CHANGE IS $x \to xc$ AND $v \to v$. NOW IF WE USE $\xi = \ln x$, ξ SCALES AS $x + \ln c$ AND v SCALES AS v. THE FORM OF THE DIFFERENTIAL EQUATION IS NO

$$\frac{dv}{d\xi} = f(v)$$

AND ONE VARIABLE HAS BEEN REMOVED; IN THIS CASE ξ.

AS AN EXAMPLE $y = vx$, $\dfrac{dy}{dx} = \dfrac{dv}{dx}\,x + v = \dfrac{dv}{d\xi} + v$

$$\frac{dv}{d\xi} + v = v^2 + 1 \longrightarrow \frac{dv}{v^2 - v + 1} = d\xi$$

AS ANOTHER EXAMPLE

$$\frac{dy}{dx} = x + \frac{x^3}{y} \qquad y \text{ GOES AS } x^2 \text{ SO IF } \begin{matrix} x \to cx \\ y \to c^2 y \end{matrix}$$

THEN $y = x^2 v$ AND $\xi = \ln x$

$$\frac{d\,x^2 v}{dx} = 2xv + x^2\frac{dv}{dx} = x + \frac{x^3}{x^2 v}$$

$$2v + \frac{dv}{d\xi} = 1 + \frac{1}{v}$$

The linear 1^{st} order differential equation has the general form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

And has the general solution

$$y = e^{-F(x)} \int_A^x e^{F(x')}\, Q(x')\, dx' + A e^{-F(x)}$$

Where $\quad F(x') = \int^x P(x'')\, dx''$

For $Q=0$, the homogeneous case the special solution is just

$$y = A e^{-F(x)}$$

Once this equation is solved we can also solve

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

which is rewritten as

$$\frac{dy}{y^n dx} + P(x) \frac{1}{y^{n+1}} = Q(x)$$

A special case is called the Riccati equation

$$\frac{dy}{dx} + a(x)\, y^2 = b(x)$$

This cannot in general be solved but it is unusual what you can do with it. We can rewrite it as

$$\frac{1}{a}\frac{dy}{dx} + y^2 = \frac{b(x)}{a(x)}$$

Now let $\xi = \int^x a(x')\, dx'$ so that $d\xi = a(x)\, dx$

Then we have

$$\frac{dy}{d\xi} + y^2 = g(\xi)$$

Now let $y = \frac{1}{\psi}\frac{d\psi}{dx} = \frac{\psi'}{\psi}$, don't ask why you do this just do it! Then we can write

$$\frac{dy}{dx} = \frac{\psi''}{\psi} - \frac{(\psi')^2}{\psi^2} \longrightarrow \frac{dy}{dx} + y^2 = \frac{\psi''}{\psi} = g(\xi)$$

Which is written as

$$\frac{d^2\psi}{d\xi^2} = g(\xi)\, \psi$$

This is now linear in ψ and soluable but the order is increase to second. Since the final equation is like the wave equation, the solution is more familiar to us.

PROPERTIES of HIGHER ORDER EQUATIONS

WE HAVE ALREADY DISCUSSED LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS AND FOUND THEY CAN ALWAYS BE SOLVED. THE CASE OF NONLINEAR DIFFERENTIAL EQUATIONS IS ANOTHER MATTER WHICH WE WILL DISCUSS IN THE FOLLOWING CASES.

CASE 1. NO Y. $F(y'', y', y, x) = 0$
REDUCE ORDER BY LETTING $w = y'$

CASE 2. NO X. USE dx/dy SAME AS BEFORE
TRY #1

CASE 3. EQUATION HAS HOMOGENITY, P, $P\frac{dP}{dy}$, $P\frac{dP^2}{dy^2}$

CASE 4. $y'' = f(y)$ MULTIPLY BY y' THEN INTEGRATE

$$\frac{d}{dx}\left(\frac{y'^2}{2}\right) = F(y)\,dy$$

CASE 5 $y'' + f(x)y' + \varphi(x)y'^2 = 0$
DIVIDE OUT y'

LINEAR DIFF EQ IN y BUT HIGHER ORDER.

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots P_1(x)\frac{dy}{dx} + P_0(x)y = Q(x)$$

IF $Q = 0$ THE HOMOGENEOUS SOLUTION CAN BE USED TO FIND THE NONHOMOGENEOUS PART.

SOLVING DIFFERENTIAL EQUATIONS NUMERICALLY

SOLVING INTEGRAL EQUATIONS

A EXAMPLE OF A LINEAR HOMOGENEOUS INTEGRAL EQUATION IS THE FOLLOWING

$$f(x) = \lambda \int_0^\infty e^{-y} \left[f(x+y)\right]^2 dy$$

HERE THE INTEGRAL CANNOT BE DONE SINCE IT INVOLVES $f(x+y)$ WHICH IS WHAT WE ARE TRYING TO FIND. TYPICALLY THEN INTEGRAL EQUATIONS ARE SOLVED BY THE METHOD OF ITERATION. THIS IS A MATHEMATICAL WAY OF SAYING TRIAL AND ERROR. YOU HAVE TO GUESS A $f(x)$ THEN PLUG IN TO THE INTEGRAL, EVALUATE THE RECALCULATE $f(x)$ AND TRY AGAIN.

THE IS A SMALL CLASS OF INTEGRAL EQUATIONS WHICH WE CAN GET RID OF RIGHT AWAY. THIS INVOLVES AN EQUATION OF THE FORM

$$f(x) = \lambda \int K(x,y)\, f(y)\, dy \quad + g(x)$$

HERE $K(x,y)$ IS CALLED THE KERNEL. AN EXAMPLE WOULD BE

$$f(x) = \int e^{-(y-x)} f(y)\, dy$$

THIS IS REFERRED AS THE DISPLACEMENT KERNEL, $K(X-Y)$. NOW THE EASIEST WAY TO PROCEED IS TO TAKE THE CONVOLUTION OF THIS INTEGRAL. THAT IS,

$$\int e^{ikx} f(x)\, dx = \varphi(x) = \lambda \int_{-\infty}^{\infty} e^{-(y-x)} e^{ik(x-y)} e^{iky} f(y)\, dy$$

OR

$$\varphi(k) = \lambda\, w(k)\, \varphi(k)$$

WHERE

$$W(k) = \int e^{-|u|} e^{iku}\, du = \frac{1}{1+k^2}$$

THEN

$$\varphi(k) = \frac{e^{-k^2/2}}{1-\lambda w(k)}$$

ANOTHER EXAMPLE IS

$$f(x) = \int_0^1 (1-xy) f^3(y)\, dy$$

$$= \int_0^1 f^3(y)\, dy - x\int_0^1 y f^3(y)\, dy$$

$$\equiv C - xD$$

WHERE

$$C = \int_0^1 (C-Dy)^3\, dy$$

$$D = \int_0^1 y(C-Dy)^3 dy$$

ANOTHER EXAMPLE IS

$$f(x) = \int_0^\infty \frac{\cos(x-y)}{1+f(y)^2}\, dy$$

$$= \cos x \int \frac{\cos y}{1+f(y)^2}\, dy + \sin x \int \frac{\sin y}{1+f(y)^2}\, dy$$

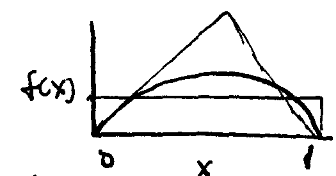
$$= A\cos x + B \sin x$$

WHERE A AND B MUST BE NUMERICALLY SOLVED AND THE SOLUTION ITERATED.

CALCULUS of VARIATIONS

Now I WANT TO SOLVE SOME PROBLEMS of A DIFFERENT TYPE. AS EXAMPLE SUPPOSE I HAVE A STRING AND I WANT TO ENCLOSE THE MAXIMUM AREA. WHAT SHAPE DOES THE STRING TAKE? THE AREA IS GIVEN AS

$$A = \int_0^1 f(x)\, dx$$



A IS THEN A NUMBER WHICH DEPENDS ON A FUNCTION. WE CALL A A FUNCTIONAL SINCE IT DEPENDS ON f(x).

AS ANOTHER EXAMPLE SUPPOSE I THROW A BALL UP IN THE AIR. WHAT PATH WILL IT FOLLOW? THE ANSWER IS THE ONE THAT MAKES THE TOTAL ENERGY LEAST. BUT WHICH ONE IS THAT? WE WILL DEFINE A NEW TERM A, THE ACTION, WHICH IS THE TIME INTEGRAL of KE−PE. THUS

$$A = \int \left[\frac{m}{2}\left(\frac{dh}{dt}\right)^2 - mg\,h(t)\right] dt$$

THE PROBLEM IS TO FIND THE CURVE WHICH MAKES THIS ACTION MINIMUM. TO DO THIS WE MUST MAXIMIZE THE INTEGRAND. THIS MEANS THAT IF $f(\bar{X})$ DESCRIBES THE ACTUAL PATH of THE BALL THEN $f(\bar{X}+\epsilon)$ IS 2nd ORDER IN ϵ, I.E. YOU CAN GO OFF THE PEAK of THE HILL IN ANY DIRECTION AND NOT CHANGE YOUR SLOPE.

SUPPOSE THEN $\hbar(t)$ IS THE FUNCTION WHICH MAXIMIZES THE INTEGRAND. NEXT TRY $h(t) = \hbar(t) + \eta(t)$ WHICH IS SLIGHTLY DIFFERENT FROM THE CORRECT PATH. THEN OUR CONDITION BECOMES

$$A[h(t)] = A[\hbar(t)] + \text{No 1st order in } \eta$$

SUBSTITUTING THEN h(t), WE HAVE

$$A[h(t)] = \int \left\{ \frac{m}{2}[\dot{\hbar} + \dot{\eta}]^2 - mg(\hbar + \eta)\right\} dt$$

EXPANDING

$$= \int \left[\frac{m}{2}\dot{\hbar}^2 - mg\hbar + m\dot{\hbar}\dot{\eta} - mg\eta + \underbrace{\cancel{\frac{m}{2}\dot{\eta}^2}}_{0} \right] dt$$

WE NOW WANT

$$\int \left[\frac{m}{2}\dot{\hbar}^2 - mg\hbar(t)\right] dt + \text{no 1st order in } \eta$$

Thus we require

$$\int_0^t [m\dot{h}\dot{\eta} - mg\eta]\, dt =$$

which can be integrated by parts

$$= m\dot{h}\,[\cancel{\eta(t)}]_0^t - \int_0^t m\ddot{h}(t)\,\eta(t) - \int mg\eta(t)\,dt$$

$$= \quad 0 \quad - \int_0^t (m\ddot{h}(t) + mg)\,\eta(t)\,dt$$

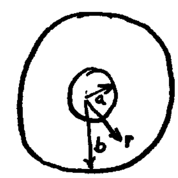
Then we have

$$\ddot{h}(t) = -g$$

This is the expected result of a particle ~~in~~ falling in a gravitational field.

MORE ON THE VARIATION PRINCIPLE

I MENTIONED LAST TIME how THE VARIATIONAL PRINCIPLE WAS USEFUL IN SOLVING PROBLEMS. I'd LIKE TO ILLUSTRATE THIS by WORKING OUT ANOTHER EXAMPLE THIS TIME IN ELECTROSTATICS. CONSIDER TWO CYLINDRICAL CONDUCTOR which ARE CONCENTRIC. THE INNER of RADIUS a IS AT A POTENTIAL V WHILE THE OUTER of RADIUS b IS AT 0 POTENTIAL. THE POTENTIAL φ BETWEEN THE CONDUCTORS CAN be QUITE COMPLICATED DEPENDING ON THE SURFACE CHARGE HOWEVER THE EXACT φ IS THE ONE WHICH MINIMIZES THE ENERGY, I.E WHICH MAKES THE ENERGY INTEGRAL MINIMUM



$$E = \frac{\epsilon_0}{2} \int (\nabla\varphi)^2 \, d\,Vol = \text{MINIMUM}$$

THIS ENERGY IS THE ENERGY of THE SYSTEM WHICH IS EQUAL TO $\frac{1}{2} CV^2$ WHERE C IS THE CAPACITANCE. THUS SINCE V IS FIXED WE CAN FIND C.

THE CORRECT ANSWER FOR φ WE KNOW; IT IS A $1/r$ VARIATION SUCH THAT THE EXACT VALUE FOR C IS

$$C = \frac{2\pi\epsilon_0}{\ln b/a}$$

WE MIGHT SEE HOW CLOSE WE CAN GET THIS ANSWER by TAKING A TRIAL φ WHICH IS NOT THE CORRECT ONE. LETS FIRST TRY A LINEARLY DECREASING FIELD,

$$\varphi = V \frac{(1 - r/b)}{(1 - a/b)}$$

SUBSTITUTING AND INTEGRATING WE GET

$$\frac{C_{LIN}}{2\pi\epsilon_0} = \frac{b+a}{2(b-a)}$$

ANOTHER GUESS MIGHT be A QUADRATIC FUNCTION

$$\varphi = V \left[1 + \alpha \frac{(r-a)}{(b-a)} - (1+\alpha)\left(\frac{r-a}{b-a}\right)^2 \right]$$

OUR PROBLEM IS TO SELECT THE BEST CURVATURE OR α FROM THE FAMILY of PARABOLAS.

WE'LL PROCEED BY COMPUTING THE INTEGRAL

$$\nabla\varphi = V\left[\alpha - 2(1-\alpha)\frac{(r-a)}{(b-a)}\right]$$

$$\nabla\varphi^2 = V^2\left[\alpha - 2(1-\alpha)\frac{(r-a)}{(b-a)}\right]^2$$

$$E = \frac{\epsilon_0}{2}\int_a^b V^2\left[\alpha - 2(1-\alpha)\frac{(r-a)}{(b-a)}\right]^2 2\pi r\,dr$$

INTEGRATING AND SOLVING FOR C

$$\frac{C}{2\pi\epsilon_0} = \frac{a}{b-a}\left[\frac{b}{a}\left(\frac{\alpha^2}{6} + \frac{2\alpha}{3} + 1\right) + \frac{1}{6}\alpha^2 + \frac{1}{3}\right]$$

NOW I NEED TO PICK THE α WHICH ~~MINIMIZES~~ MINIMIZES THIS FUNCTION. THUS TO FIND C_{MIN} I DIFFERENTIATE C WRT α AND SET EQUAL TO 0. WHEN I DO THAT I FIND

$$\alpha = -\frac{2b}{b+a}$$

AND FINALLY I HAVE

$$\frac{C_{PAR}}{2\pi\epsilon_0} = \frac{b^2 + 4ab + a^2}{3(b^2-a^2)}$$

NOW LET ME COMPARE THESE TWO RESULTS WITH THE ACTUAL VALUE FOR $C/2\pi\epsilon_0$ FOR DIFFERENT RATIOS OF b/a.

$\frac{b}{a}$	$\frac{C_{TRUE}}{2\pi\epsilon_0}$	$\frac{C_{LINEAR}}{2\pi\epsilon_0}$	$\frac{C_{QUAD}}{2\pi\epsilon_0}$
2	1.4423	1.5	1.446
4	.721	.833	0.733
10	.434	.612	.475
100	.267	.51	.346
1.5	2.4662	2.50	2.4667
1.1	10.492070	10.500000	10.492065

THUS FOR SMALL DIFFERENCES IN b AND a THE TWO TRIALS WORK WELL WITH THE QUADRATIC TRIAL BEING EXCEPTIONAL. IT IS ONLY WHEN THE RATIO GETS UP TO 100 TO 1 THAT THE QUADRATIC MODEL BREAKS DOWN

THE VARIATIONAL PRINCIPLE finds MANY APPLICATION IN IN PHYSICS. IT IS VERY POWERFUL WHEN THE PROBLEM INVOLVES NO LOSSES. MANY PROBLEMS HAVE MINIMUM PRINCIPLES AS THEIR BASIS.

YOU MIGHT BE INTERESTED TO KNOW THAT THE MAXWELL EQUATIONS for free SPACE CAN be derived from A MINIMUM PRINCIPLE GIVEN A VECTOR POTENTIAL $\bar{A}(\bar{x},t)$ AND A SCALAR POTENTIAL $\varphi(x,t)$. THE ACTION IS MINIMUM for THE SYSTEM which obeys MAXWELL'S EQUATIONS. THUS WE MAY WRITE

$$S = \int \left\{ (\nabla A_x)^2 - \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} + (\nabla A_y)^2 - \frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2} + (\nabla A_z)^2 - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} - \nabla^2\varphi \frac{1}{c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 \right\} dV dt$$

$$\cancel{\nabla^2\varphi + \frac{1}{c}} \quad + \int [(\vec{A}\cdot\dot{\vec{j}}) - \varphi\rho]\, dV dt$$

OTHER MINIMUM PRINCIPLE'S INVOLVE LIGHT which TAKES THE MINIMUM TIME BETWEEN TWO POINTS - FERMAT'S PRINCIPLE. Shroedinger's EQUATION ALSO obeys A MINIMUM ENERGY PRINCIPLE:

$$E_{MIN} = \int \left[\frac{\hbar^2}{2m} |\nabla\psi|^2 + V(x,y,z)\,\psi^*\psi \right] dVol$$

E IS MINIMUM for ALL NORMALIZED WAVE functions, i.e. ψ's which SATISFY $\int \psi^*\psi^* \, dVol = 1$

MATRICES

A matrix is an array of numbers layed out in a rectangle shape which satisfy certain combination laws and relationships. A special class of matrices is the square matrix which we'll deal with.

The matrix is layed out in a row column network where i denotes the row number and j denotes the column number. The each element is denoted by an a_{ij}. The index i and j both run to N so the matrix is dimensional NXN. The array is denoted as $\underline{a}$.

Now two arrays can be added, i.e

$$\underline{a} + \underline{b} = \underline{c}$$

If $a_{ij} + b_{ij} = c_{ij}$

The product of two matrices is given as

$$\underline{a}\,\underline{b} = \underline{c}$$

where
$$c_{ij} = \sum_k a_{ik}\, b_{kj}$$

As an example of how this multiplication works let's multiply the following two matrices together

$$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} \begin{vmatrix} 7 & 4 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 1\times 7 + 2\cdot 5 & 1\times 4 + 2\cdot 5 \\ 4\cdot 7 + 3\cdot 5 & 4\cdot 4 + 3\cdot 6 \end{vmatrix} = \begin{vmatrix} 17 & 16 \\ 43 & 34 \end{vmatrix}$$

You always go across $\underline{a}$ and down $\underline{b}$ when multiplying. For practise take the following four matrices and show the following

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then
$$\sigma_x^2 = \sigma_y^2 = \sigma_3^2 = 1^2 = 1 \quad ; \quad \sigma_x \sigma_y = i\sigma_3 = -\sigma_y \sigma_x$$

In matrix multiplication the product $\underline{a}\,\underline{b}$ is different from the product $\underline{b}\,\underline{a}$. Thus you have to be careful what your doing.

An example of matrix multiplication is linear coordinate transformation. Suppose we have the set of coordinates $x_1, x_2, x_3, \ldots x_n$ which for a $1 \times N$ rectangular matrix. More commonly a $1 \times N$ matrix is called a vector. Now a new set of coordinates can be written as a linear sum of the old coordinates,

$$
\begin{aligned}
x_1' &= a_{11} x_1 + a_{12} x_2 + \cdots \\
x_2' &= a_{21} x_1 + a_{22} x_2 + \cdots \\
&\vdots \\
x_n' &= a_{n1} x_1 + \cdots \cdots a_{nn} x_n
\end{aligned}
$$

This is a square matrix where

$$x_j' = \Sigma a_{ji} x_i \Rightarrow \vec{\underline{x}'} = \underline{a} \vec{x}$$

Thus we can think of multiplying a vector by a matrix to produce a new vector. If we made another transformation again say

$$x'' = \Sigma b_{ji} x_i'$$

The final transformation involves two transformation

$$x'' = \sum_i \left(\sum_k b_{jk} a_{ki} \right) x_i = \Sigma c_{ji} x_i$$

The combine transformation can be expressed as

$$\underline{c} = \underline{b}\,\underline{a}$$

The order being first a transformation by $\underline{a}$ then one by $\underline{b}$. Another way of writing this is

$$D(R_3) = D(R_2) D(R_1)$$

Proceeding on in definitions and terminology. The unit matrix is defined to be

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 = I$$

The unit matrix has the property that

$$\underline{I}\,\underline{a} = \underline{a} = \underline{a}\, I$$

Thus $\underline{I}$ is said to commute. It follows that

$$\Sigma\, \delta_{ik} a_{kj} = a_{ij}$$

MULTIPLICATION OF A MATRIX BY A NUMBER INVOLVES MULTIPLYING EACH ELEMENT BY THAT NUMBER, I.E

$$\beta \underline{a} = \beta a_{ij}$$

THE ZERO MATRIX IS THE NULL MATRIX:

$$\underline{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

THE RECIPROCAL OF A MATRIX IS CALLED THE INVERSE MATRIX

$$X_i' = \sum_j a_{ij} X_j \longrightarrow X_i = \sum \alpha_{ij} X_j'$$

$\underline{\alpha}_{ij}$ IS THE RECIPROCAL OR INVERSE OF $\underline{a}$ AND IS DENOTED AS a^{-1}. IT IS TRUE THAT $\underline{\alpha}\ \underline{a} = 1 = \underline{a}^{-1}\underline{a}$

MATRIX MULTIPLICATION IS ASSOCIATIVE, I.E.

$$\underline{a}(\underline{b}\,\underline{c}) = (\underline{a}\,\underline{b})\underline{c}$$

MATRIX ADDITION IS DISTRIBUTIVE

$$\underline{a}(\underline{b}+\underline{c}) = \underline{a}\,\underline{b} + \underline{a}\,\underline{c}$$

MATRIX MULTIPLICATION IS COMMUTATIVE IF

$$\underline{a}\,\underline{b} = \underline{b}\,\underline{a}$$

THE COMMUTATOR OF $\underline{a}$ AND $\underline{b}$ IS DEFINED TO BE

$$\underline{a}\,\underline{b} - \underline{b}\,a = [\underline{a}, \underline{b}]$$

THE TRANSPOSE OF $\underline{a}$ IS

$$\underline{a}^{\dagger} = (a^{\dagger})_{ij} = \alpha_{ji}$$

THE TRANSPOSE INTERCHANGES THE ROWS AND COLUMNS WHILE LEAVING THE MAIN DIAGONAL UNCHANGED.

THE HERMITIAN ADJOINT, $\underline{a}^{*}$ IS DEFINED AS

$$(a^{*})_{ij} = a_{ji}^{*} = \text{COMPLEX CONJUGATE OF } a_{ji}$$

A REAL MATRIX HAS REAL ELEMENTS;

$$\underline{a} = \underline{a}^{*}$$

A SYMMETRIX MATRIX IS ONE WHICH EQUALS ITS TRANSPOSE

$$\underline{a} = \underline{a}^{\dagger}$$

A HERMITIAN MATRIX IS ONE WHICH EQUALS THE COMPLEX CONJUGATE OF ITS TRANSPOSE

$$\underline{a} = \underline{a}^{T*}$$

A UNITARY MATRIX SATISFIES

$$\underline{a}^{*} = \underline{a}^{-1}$$

TWO FACTS WORTH REMEMBERING

$$(ab)^{-1} = \underline{b}^{-1}\underline{a}^{-1}$$

AND

$$(\underline{a}\,\underline{b})^{*} = \underline{b}^{*}\underline{a}^{*}$$

THE DETERMINATE OF A MATRIX IS THE PRODUCT SUM OF THE ELEMENTS $\det A$

THE TRACE OF A MATRIX IS THE SUM OF THE DIAGONAL ELEMENTS

$$\mathrm{Tr}\,\underline{a} = \sum_i a_{ii}$$

THE TRACE HAS THE FOLLOWING PROPERTIES

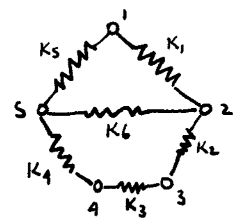
$$\mathrm{Tr}\,(\underline{a}\,\underline{b}) = \mathrm{Tr}\,(\underline{b}\,\underline{a})$$

$$\mathrm{Tr}\,(\underline{a}+\underline{b}) = \mathrm{Tr}\,(\underline{b}+\underline{a})$$

$$\mathrm{Tr}\,(\underline{a}\,\underline{b}\,\underline{c}) = \mathrm{Tr}(\underline{c}\,\underline{a}\,b) \neq \mathrm{Tr}\,(\underline{a}\,\underline{c}\,\underline{b})$$

APPLICATION OF MATRIX THEORY TO SOLVING THE POLYATOMIC MOLECULE

WE HAVE BEEN DISCUSSING THE THEORY OF MATRICES NOW LETS APPLY WHAT WE HAVE LEARNED TO SOLVING A SPECIFIC PROBLEM. SUPPOSE I HAD A MOLECULE OF SOME SHAPE AND TO FIRST APPROXIMATION WE CAN IMAGINE THIS MOLECULE TO CONSIST OF A NUMBER OF DISCRETE MASSES AND SPRINGS.



EACH MASS IS DECRIBED BY 3 COORDINATES $x_1, y_1, z_1, x_2, y_2, z_2, \cdots$ SUCH THAT THE KINETIC ENERGY OF THE SYSTEM IS GIVEN BY

$$K.E = \tfrac{1}{2} m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2) + \tfrac{1}{2} m_2(\dot{x}_2+\dot{y}_2+\dot{z}_2)^2 + \cdots$$

THE POTENTIAL ENERGY IS GIVEN BY

$$V = V(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)$$

NOW I WANT TO DESCRIBE THE SYSTEM BY A NEW SET OF COORDINATES WHICH I SHALL DEFINE TO BE:

$$q_1 = \sqrt{m_1}\, x_1 \qquad q_2 = \sqrt{m_2}\, x_2 \qquad q_3 = \sqrt{m_3}\, x_3 + \cdots$$

WITH THESE NEW VARIABLES WE MAY WRITE

$$KE = \sum_i \tfrac{1}{2} \dot{q}_i^2 \qquad PE = V(q_1, q_2, \ldots)$$

IF THE SYSTEM IS IN EQUILIBRIUM, THE ENERGY IS A MINIMUM. IF THIS STATE IS DEFINED BY THE SET OF VARIABLES $\bar{q}$ THE CONDITION WHICH MUST BE SATISFIED IS $\partial V/\partial q\,|_{\bar{q}} = 0$. THUS WE CAN EXPAND ABOUT THIS EQUILIBRIUM STATE FOR SMALL PERTURBATIONS, I.E. LET $q = \bar{q} + \epsilon$. EXPANDING V AS A POWER SERIES GIVES

$$V = V(\bar{q}_1, \bar{q}_2, \cdots) + \sum_i (q_i - \bar{q}_i)\frac{\partial V}{\partial q_i}\Big|_{\bar{q}_i} + \frac{1}{2}\sum_{ij}(q_i - \bar{q}_i)^2 \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\bar{q}}$$

$$+ \text{higher order terms}$$

NOW ALL FIRST ORDER TERMS $= 0$. AND ALL WE RETAIN IS THE ZEROTH ORDER AND SECOND ORDER TERM. IF WE REDEFINE OUR VARIABLES FROM THE EQUILIBRIUM, $\bar{q}$, I.E $q_i' = q_i - \bar{q}_i$ THEN ONLY THE QUADRATIC TERM REMAINS.

I CAN SIMPLIFY V BY DEFINING THE NEW QUANTITY, C_{ij}, SUCH THAT

$$C_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

THE ENERGY OF THE SYSTEM CAN NOW BE WRITTEN AS

$$E = \frac{1}{2}\sum_i \dot{q}_i^2 + \frac{1}{2}\sum_i \sum_j C_{ij} q_i q_j$$

THE SYSTEM HAS BEEN REDUCED TO A SET OF PARTICLES AND INTERCONNECTING SPRINGS. C_{ij} REPRESENTS THE INTERACTING FORCES BETWEEN PARTICLES. WITH THE TOTAL ENERGY WE CAN SOLVE FOR THE EQUATION OF MOTION BY ~~THE~~ FINDING THE MOTION WHICH MINIMIZES THE ACTION. TO ACCOMPLISH THIS WE WRITE THE LAGRANIAN, L, FOR THE SYSTEM,

$$L = KE - PE = \frac{1}{2}\sum \dot{q}_i^2 - \frac{1}{2}\sum C_{ij} q_i q_j$$

NOW WE WANT TO MINIMIZE THIS FUNCTION, I.E

$$\int \left(\frac{1}{2}\sum \dot{q}_i^2(t) - \frac{1}{2}\sum_{ij} C_{ij} q_i q_j \right) dt = \text{MIN}$$

TO PROCEED LET $q_i = \bar{q}_i + \eta_i$ WHERE $\bar{q}_i$ IS THE CORRECT MOTION OF THE i^{th} PARTICLE AND η_i IS ITS PERTURBATION FROM THIS MOTION. DIFFERENTIATING AND SUBSTITUTING WE GET TO FIRST ORDER IN η

$$-\int \left(\sum_i \ddot{q}_i \eta_i + \sum_{ij} C_{ij} q_j \eta_i \right) dt = \text{MIN}$$

FOR THIS INTEGRAL TO BE A MINIMUM THE FOLLOWING CONDITION MUST BE MET

$$-\ddot{q}_i(t) = \sum C_{ij} q_j(t)$$

THIS CONDITION IS A SET OF EQUATIONS OF MOTION WHICH SAYS THE ACCELERATION OF THE i^{th} PARTICLE IS DUE TO ~~ALL~~ THE SUM OF ALL THE OTHER FORCES ACTING ON IT. IF i RUN FROM 1 TO N, THERE ARE N DIFFERENTIAL EQUATIONS.

WE NOW WANT TO DISCUSS THE SOLUTION TO THESE EQUATIONS IN MATRIX NOTATION. THE q'S ARE REPRESENTED BY A COLUMN MATRIX, SOMETIMES CALLED A VECTOR,

$$\bar{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

The symbol C_{ij} stands for a normal matrix, $\underline{C}$. In our new notation we may simply the equations of motion to

$$-\ddot{\bar{q}}_i = \underline{C}\,\bar{q}_i$$

Thus the solution has the property that its second derivative is proportional to itself. We may guess a solution since we have seen this equation ~~equ~~ before. We know by starting the motion in a certain way all the disturbances will respond at characteristics frequencies. The characteristic motion of the system is described by its normal modes. These normal modes, $\bar{u}_i$, will be the solution to the above equation, i.e. the satisfy the equality

$$\bar{q}_i = \bar{u}_i\, e^{i\omega t}$$

Now we will pursue the analysis in matrix notation but let me caution that the only virtue of this approach is one of expediency. It says you paper but it doesn't do anything else. You don't understand the problem any easier; in fact if you are not careful the notation will confuse you. Thus using our matrix notation we may write the equations of motion as

$$-\omega^2\, \bar{u}_i = \underline{C}\, \bar{u}_i \equiv \lambda\, \bar{u}_i$$

The parameter λ is refered to as the eigenvalue or eigen-frequencies of the motion. Now we need the $\bar{u}_i$'s which tells us the size of disturbance of the i^{th} mode. Thus we need to solve this matrix / vector equation

$$\underline{C}\,\bar{u}_i = \lambda\,\bar{u}_i$$

This can be written as

$$\Sigma_j\, C_{ij} u_j = \lambda u_i$$

The problem is to solve a set of linear equations for N unknowns (the u_N's). It turns out that there are not N independent variables since the following equation must hold

$$\Sigma_j\, (C_{ij} - \lambda \delta_{ij})\, u_j = 0$$

or defining the new matrix $\underline{A} = \underline{C} - \lambda \underline{I}$ it follows that $A^{-1} = \infty$ since $\underline{A}\,\bar{u}_j = 0$

The only solution to $\underline{A}\,\bar{U} = 0$ other than the trivial case of $\bar{U} = 0$ is if $\underline{A}$ is singular which implies that its determinet is 0. Thus we can only get a vibration

if $\quad \det \underline{C} - \lambda \underline{I} = 0$

To establish which λ's satisfy this equality requires working out the determinant. $\underline{C}$ is given by the dynamical equations and therefore known. Thus we must work out the determinant of

$$\begin{pmatrix} C_{11}-\lambda_1 & C_{12} & C_{13} & \cdots & C_{1N} \\ C_{21} & C_{22}-\lambda_2 & C_{23} & \cdots & C_{2N} \\ \vdots & & & & \\ C_{n1} & \cdots & \cdots & & C_{nn}-\lambda_n \end{pmatrix}$$

At most there are N different values for λ. However there may be less if several are equal or degenerate. To find the λ's we must solve a polynomial of degree N. We want the roots of the polynomial so we set it ~~equate~~ to zero.

We shall first consider the case where all the roots are different, we shall denote the roots as $\lambda^{(r)}$ where r goes from 1 to N. Therefore the solutions u's will go with a particular $\lambda^{(r)}$, i.e, for each $\lambda^{(r)}$ there is a $\bar{u}^{(r)}$. This says for each frequency there is a characteristic motion or pattern of vibration. The equations which are then satisfied are

$$\underline{C}\,\bar{u}_i^{(r)} = \lambda^{(r)}\,\bar{u}_i^{(r)}$$

or

$$\sum C_{ij}\,u_j^{(r)} = \lambda^{(r)}\,u_i^{(r)}$$

Now we'll look at some of properties of this solution. First the solutions are orthogonal. This means mathematically that

$$\sum_i u_i^{(r)}\,u_j^{(s)} = 0$$

This is true only if $r \neq s$, i.e. $\lambda^{(r)} \neq \lambda^{(s)}$. To prove this we show that

$$\sum C_{ij}\,u_j^{(s)} = \lambda^{(s)}\,u_i^{(s)}$$

$$\sum_{ij} u_i^{(s)} C_{ij}\,u_j^{(r)} = \lambda^{(r)} \sum u_i^{(r)} u_i^{(s)}$$

$$\sum_{ij} u_i^{(r)} C_{ij}\,u_j^{(s)} = \lambda^{(s)} \sum u_i^{(r)} u_i^{(s)}$$

Now $C_{ij} = C_{ji}$ and $\lambda^r - \lambda^s \sum u_i^{(r)} u_i^{(s)} = 0$. Since $\lambda^r \neq \lambda^s$ it must follow that $\sum u_i^r u_i^s = 0$. This means $\bar{u}^r$ and $\bar{U}^s$ are perpendicular to eachother.

THE PROPERTY OF ORTHOGONALITY IS VERY USEFUL AS YOU WILL LATER SEE.

NOW ANOTHER PROPERTY OF THE SOLUTION WE WILL FIND USEFUL IS THE NUMERICAL VALUE OF $\sum_i u_i^{(n)} u_i^{n}$. IT TURNS OUT WE CAN ONLY GET THE A PROPORTIONAL SIZE OF THE u'S. WE DON'T HAVE ENOUGH INFORMATION TO GET THE ABSOLUTE SIZE OF THE $\bar{u}$'S. TO ESTABLISH A SOLUTION WE CHOOSE THE NORMALIZING CONSTRAINT THAT

$$\sum u_i^{(n)} u_i^{(n)} = 1$$

AS A SUMMARY OF THE ORTHOGONALITY AND NORMALITY CONDITIONS ON THE u'S WE MAY WRITE IN THE SHORTHAND NOTATION

$$(\bar{u}_i^{(n)}, \bar{u}_i^{(s)}) = \delta_{ns}$$

WHERE $\delta_{ns} = 0$ IF $n \neq s$ AND $= 1$ IF $n = s$. THIS NOTATION IS CALLED THE INNER OR SCALAR PRODUCT OF TWO VECTORS.

THE SOLUTION WE HAVE OBTAINED IS MORE GENERAL THAN IT APPEARS. THE GENERALITY INVOLVES THE SOLUTION OF THE PARTICLE MOTION WHERE THE DISTURBANCE DOES NOT QUITE RESONATE WITH A NORMAL. IF WE DESCRIBE THE INITIAL CONDITION OF SYSTEM ~~AND~~ AT $t=0$ AS $q_i(0)$, THEN WE CAN FIND THE SUBSEQUENT MOTION $q_i(t)$. THE REASON WE CAN DO THIS IS BECAUSE WE CAN ADD TOGETHER THE SOLUTIONS WITH ARBITRARY COEFFICIENTS, E.G.

$$q_i(t) = a^{(1)} u^{(1)} e^{i\omega^{(1)} t} + a^{(2)} u^{(2)} e^{i\omega^{(2)} t}$$

OR

$$q_i(t) = \sum_n a^{(n)} \bar{u}_i^{(n)} e^{i\omega^{(n)} t}$$

HERE THE $u_i^{(n)}$ ARE THE NORMAL MODES OF THE SYSTEMS WHICH HAVE BEEN WORKED OUT. THE $a^{(n)}$ ARE ~~THE~~ NUMBERS WHICH DEPENDS ON ~~HOW~~ THE SYSTEM IS STARTED. THAT IS IT TELLS US HOW MUCH OF EACH NORMAL MODE IS PRESENT. THE $a^{(n)}$ CANNOT BE COMPUTED AHEAD OF TIME. THE INTERESTING THING IS GIVEN $q_i(0)$ THE $a^{(n)}$'S CAN BE FOUND. THE INITIAL CONDITIONS CAN BE EXPRESSED AS

$$q_i(0) = \sum a^{(n)} u_i^{(n)}$$

IT first LOOKS LIKE THINGS ARE GOING from BAD TO WORSE. BUT IF WE RECALL A SIMILAR PROBLEM WHEN DEALING WITH FOURIER SERIES WE MAY GET SOMEWHERE. THERE WE WANTED TO KNOW HOW MUCH EACH MODE ADDED TO GIVE A COMPOSITE SIGNAL. WE WERE ADDING SINES OR COSINES OF DIFFERENT FREQUENCIES. IN ORDER TO DETERMINE HOW MUCH OF ONE MODE IS PRESENT WE MAY MULTIPLY ~~THAT~~ BY ANOTHER MODE, I.E.

$$\sum_i u_i^{(s)} q_i(0) = \sum_n a^{(n)} \sum_i u_i^{(s)} u_i^{(n)}$$

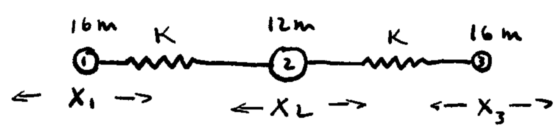
NOW THE RIGHT SIDE $= 0$ UNLESS $n = s$ SO ONLY $a^{(s)}$ SURVIVES AND WE HAVE

$$a^{(s)} = \sum u_i^{(s)} q_i(0) = (\bar{u}_i(0), \bar{q}_i(0))$$

THUS WE HAVE A GENERALIZATION OF A FOURIER EXPANSION.

LINEAR VIBRATION OF CO_2

LET'S SEE HOW THIS ALL WORKS ON THE LINEAR VIBRATION OF A CO_2 MOLECULE. THE MOLECULE IS MODELED AS 3 PARTICLES CONNECTED BY TWO SPRINGS



THE KINETIC ENERGY IS GIVEN AS

$$K.E = \tfrac{1}{2}\, 16 \dot{x}_1^2 + \tfrac{1}{2}\, 12 \dot{x}_2^2 + \tfrac{1}{2}\, 16 \dot{x}_3^2$$

NOW 1 AND 3 DO NOT INTERACT SO THE POTENTIAL ENERGY BECOMES

$$P.E = \tfrac{1}{2} K (x_1 - x_2)^2 + \tfrac{1}{2} K (x_2 - x_3)^2$$

REDEFINING VARIABLES

$$x_1 = \tfrac{1}{4} q_1 \qquad x_2 = \tfrac{1}{\sqrt{12}} q_2 \qquad x_3 = \tfrac{1}{4} q_3$$

WE HAVE

$$KE = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$$

$$PE = \tfrac{K}{2}\left[\left(\tfrac{1}{16} q_1^2\right) + \tfrac{1}{10} q_3^2 \right] + \tfrac{K}{12} q_2^2 - \tfrac{K}{4\sqrt{12}} q_1 q_2 - \tfrac{K}{4\sqrt{2}} q_2 q_3$$

The $\underline{C}$ matrix is determined as follows

$$C_{ij} = K\begin{pmatrix} \frac{1}{16} & -\frac{1}{8\sqrt{3}} & 0 \\ -\frac{1}{8\sqrt{3}} & \frac{1}{6} & -\frac{1}{8\sqrt{3}} \\ 0 & -\frac{1}{8\sqrt{3}} & \frac{1}{16} \end{pmatrix} = \frac{K}{48}\begin{pmatrix} 3 & -2\sqrt{3} & 0 \\ -2\sqrt{3} & 8 & -2\sqrt{3} \\ 0 & -2\sqrt{3} & 3 \end{pmatrix}$$

For convenience pick the spring constant $K = 48$. Now to find the λ's we must solve $\det(\underline{C} - \lambda \underline{I}) = 0$. i.e,

$$\det\begin{pmatrix} 3-\lambda & -2\sqrt{3} & 0 \\ -2\sqrt{3} & 8-\lambda & -2\sqrt{3} \\ 0 & -2\sqrt{3} & 3-\lambda \end{pmatrix} = 0$$

This is expanded out to be

$$(3-\lambda)^2(8-\lambda) - 12(3-\lambda) - 12(3-\lambda) = 0$$
$$(3-\lambda)^2(8-\lambda) - 24(3-\lambda) = 0$$

The roots of this cubic are

$$\lambda_1 = 3 \qquad \lambda_2 = 0 \qquad \lambda_3 = 11$$

Thus the frequencies of vibration are

$$\omega_1 = \sqrt{3} \qquad \omega_2 = 0 \qquad \omega_3 = \sqrt{11}$$

We now need to find the normal modes, the $u^{(\alpha)}$'s, which correspond to these particular frequencies. To do that we need to for $\underline{C}\,\bar{u}$,

$$\underline{C}\,\bar{u} = \begin{pmatrix} 3 & -2\sqrt{3} & 0 \\ -2\sqrt{3} & 8 & -2\sqrt{3} \\ 0 & -2\sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \lambda\,\bar{u}$$

This gives us three equation

$$3u_1 - 2\sqrt{3}\,u_2 + 0\cdot u_3 = \lambda u_1$$
$$-2\sqrt{3}\,u_1 \quad 0\cdot u_2 \quad - 2\sqrt{3}\,u_3 = \lambda u_2$$
$$0 \quad -2\sqrt{3}\,u_2 + 3u_3 = \lambda u_3$$

For $\lambda = \lambda' = 3$ we have

$$3u_1 - 2\sqrt{3}\,u_2 = 3u_1$$
$$-2\sqrt{3}\,u_1 + 8u_2 - 2\sqrt{3}\,u_3 = 3u_2$$
$$-2\sqrt{3}\,u_2 + 3u_3 = 3u_3$$

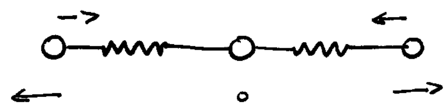
Thus we have

$$u_2^{(1)} = 0 \qquad u_1^{(1)} = -u_3^{(1)}$$

This is all we know about the proportions of elements of $\bar{u}^{(1)}$. We have only establish their relative size - not their absolute magnitude. However if we normalize $\bar{u}^{(1)}$ we get

$$\bar{u}^{(1)} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

The motion which this mode characterizes is the following



The two end atoms move out and in together such that the center carbon atoms remains stationary.

Let's now solve for the second normal mode, i.e. $\lambda^{(2)} = 0$
This time

$$3u_1 - 2\sqrt{3}\,u_2 = 0$$
$$-2\sqrt{3}\,u_2 + 3u_3 = 0$$

It follows that $u_1^{2} = u_3^{2} = 2/\sqrt{3}\; u_2$

Again normalizing

$$\bar{u}^{(2)} = a\begin{pmatrix} 2/\sqrt{3} \\ 1 \\ 2/\sqrt{3} \end{pmatrix}$$

$$a^2\left(4/3 + 1 + 4/3\right) = 1 \qquad \text{or} \qquad a = \sqrt{3/11}$$

$$\bar{u}^{(2)} = \begin{pmatrix} 2/\sqrt{11} \\ \sqrt{3}/\sqrt{11} \\ 2/\sqrt{11} \end{pmatrix} = 1/\sqrt{11}\begin{pmatrix} 2 \\ \sqrt{3} \\ 2 \end{pmatrix}$$

For this case there is no restoring force so this is the zero frequency mode wherein all the atoms move together in a translation.

For the 3rd solution $\lambda^{(3)} = 1$ we find that

$$-2\sqrt{3}\,u_2 = 8u_1$$
$$-2\sqrt{3}\,u_2 = 8u_3 \qquad u_1 = u_3$$

Which gives for $\bar{u}^{(3)}$ after normalizing

$$\bar{u}^{(3)} = \begin{pmatrix} \sqrt{3/22} \\ -4/\sqrt{22} \\ \sqrt{3}/\sqrt{22} \end{pmatrix} = \frac{1}{\sqrt{22}}\begin{pmatrix} \sqrt{3} \\ -4 \\ \sqrt{3} \end{pmatrix}$$

We can establish these modes are orthogonal by computing $(u^{(r)}, u^{(s)})$

$$(u^{(1)}, u^{(3)}) = \frac{1}{\sqrt{2}}\left(\sqrt{3/22}\right) + 0 - \frac{1}{\sqrt{2}}\sqrt{3/22} = 0$$

$$(u^{(2)}, u^{(3)}) = \frac{1}{\sqrt{22}}\,\frac{1}{\sqrt{11}}\left[2\sqrt{3} - 4\sqrt{3} + 2\sqrt{3}\right] = 0$$

$$(u^{(1)}, u^{(2)}) = \frac{1}{\sqrt{11}}\left[2/\sqrt{2} + 0 - 2/\sqrt{2}\right] = 0$$

WE SHALL NOW TRY TO SOLVE THIS PROBLEM WITH SOME INITIAL CONDITIONS. HERE THE UTILITY OF THE ORTHOGONALITY RELATIONSHIP WILL PROVE TO BE VERY USEFUL. SUPPOSE THE SOLUTION TO THE PROBLEM IS GIVEN AS

$$\bar{q}(t) = \sum_{(r)} a^{(r)} \bar{U}^{(r)} e^{i\omega^{(r)} t}$$

AND AT $t=0$ WE KNOW $\bar{q}(0)$. THE PROBLEM IS TO FIND $a^{(r)}$. WELL IT FOLLOWS THAT

$$\bar{q}(0) = \sum a^{(r)} \bar{U}^{(r)}$$

SO TO GET $a^{(r)}$ ALL WE NEED DO IS MULTIPLY BY $U^{(s)}$, I.E

$$(\bar{U}^{(s)}, \bar{q}(0)) = \sum_{(r)} a^{(r)} (U^{(s)}, U^{(r)})$$

UNLESS $s=r$ THE RHS $=0$ SO ONLY $r=s$ SURVIVES

$$(\bar{U}^{(s)}, \bar{q}(0)) = a^{(s)}$$

ANOTHER USEFUL DYNAMICAL CONCEPT IS THAT OF NORMAL COORDINATES. HERE A TRANSFORMATION IS MADE FROM THE OLD COORDINATES $\bar{q}_i$ TO A NEW SET $\bar{Q}^{(r)}$. THE NEW Q's ARE A LINEAR COMBINATION OF THE $\bar{U}_i^{(r)}$'s, I.E

$$\bar{q}_i = \sum_r Q^{(r)} \bar{u}_i^{(r)}$$

THUS, A COMPLICATED MOTION CAN BE EXPRESSED AS A LINEAR COMBINATION OF THE NORMAL MODES WHERE $Q^{(r)}$ TELLS HOW MUCH OF THE r^{th} MODE IS PRESENT. THUS AGAIN WE HAVE AN ANALOGY WITH FOURIER SERIES. IN TURN TO FIND THE $Q^{(r)}$ IN-TERMS OF THE $\bar{u}_i^{(r)}$ WE HAVE

$$Q^{(r)} = \sum_i U_i^{(r)} q_i$$

THE UTILITY IN WORKING WITH $Q^{(r)}$ IS IN SIMPLIFYING THE ENERGY EQUATION. FOR EXAMPLE THE KINETIC ENERGY CAN BE WRITTEN AS

$$KE = \tfrac{1}{2} \sum_i \dot{q}_i \dot{q}_i = \frac{1}{2} \sum_i \sum_r \sum_s \dot{Q}^{(r)} u_i^{(r)} \dot{Q}^{(s)} u_i^{(s)}$$

$\dot{Q}^{(s)}$ IS A DYNAMICAL PARAMETER AND CAN BE PULLED THROUGH $U_i^{(r)}$ SO THE SUM OVER r AND s CAN BE MADE THUS SIMPLIFYING THE EXPRESSION

$$K.E = \tfrac{1}{2} \sum \dot{Q}^{(r)} \dot{Q}^{(r)}$$

The potential energy can be simplified but requires more work,

$$PE = \frac{1}{2}\sum_{ij} q_i C_{ij} q_j$$

substituting

$$= \frac{1}{2}\sum_{r}\sum_{s}\sum_{ij} Q^{(r)} u_i^{(r)} C_{ij} Q^{s} u_j^{(s)}$$

Now

$$\sum_j C_{ij} u_j^{(s)} = \lambda^{(s)} u_i \qquad \text{so}$$

$$PE = \frac{1}{2}\sum_{r}\sum_{s}\sum_{i} \lambda^{s} Q^{r} Q^{s} u_i^{r} u_i^{(s)}$$

$$= \frac{1}{2}\sum_{r} \lambda^{(r)} Q^{(r)} Q^{(r)}$$

which is now a fairly simple expression involving only the sum of squares. Thus both the KE and PE become uncoupled equations. Therefore

$$\ddot{Q}^{(r)} = -\lambda^{(r)} Q^{(r)}$$

To show you how this normal coordinate stuff works I'll use the previous example. There

$$Q^{(1)} = \frac{1}{\sqrt{2}} q_1 + 0\, q_2 - \frac{1}{\sqrt{2}} q_2 = \frac{1}{\sqrt{2}}(q_1 - q_3)$$

$$Q^{(2)} = \frac{3}{\sqrt{11}}\left(\frac{2}{\sqrt{3}}(q_1 + q_3) + q_3\right)$$

$$Q^{(3)} = \frac{1}{\sqrt{22}}\left[\sqrt{3}(q_1 + q_3) - 4q_2\right]$$

Then

$$PE = \frac{1}{3}\, 3Q^{(1)^2} + \frac{1}{2}\, 0\, Q^{(2)^2} + \frac{1}{2}\, 11\, Q^{(3)^2}$$

Mathematically what we have done is to diagonalize the $\underline{C}$ matrix by a linear transformation.

I now want to pursue some new ideas involving matrices and vectors. Let $\underline{C} = C_{ij}$ and $\bar{u} = u_i$. The operation $\underline{C}\,\bar{u} = \bar{V}$ a new vector.

The scalar product of two vectors is

$$(\bar{x}, \bar{y}) = \sum_i x_i y_i$$

Also it can be shown that

$$(\underline{N}\,\bar{a}, \bar{B}) = (\bar{a}, \underline{N}^{\dagger}\,\bar{B})$$

I now want to talk about an equivalence transformation. This is a linear coordinate transformation of the form

$$q_1 = S_{11} q_1' + S_{12} q_2' + S_{13} q_3'$$
$$q_2 = S_{21} q_1' + S_{22} q_2' + S_{23} q_3'$$
$$\vdots$$
$$\bar{q} = \underline{S}\, \bar{q}'$$

Without any proof it is true that

$$\dot{\bar{q}} = \underline{S}\, \dot{\bar{q}}'$$

Now making this transformation in the kinetic energy equation.

$$KE = \tfrac{1}{2}(\dot{\bar{q}}, \dot{\bar{q}}) = \tfrac{1}{2}(\underline{S}\dot{\bar{q}}', \underline{S}\dot{\bar{q}}')$$

using our new fact this can be written

$$KE = \tfrac{1}{2}(\dot{\bar{q}}', \underline{S}^{+}\underline{S}\,\dot{\bar{q}}')$$

Now if we require this expression to be of the equivalent form as before i.e $\tfrac{1}{2}(\dot{\bar{q}}', \dot{\bar{q}}')$ then we require the transform must satisfy

$$S^{+}S = 1$$

or $S^{+} = S^{-1}$ which ~~mea~~ is the condition for S to be unitary.

For the PE expression we have

$$PE = \tfrac{1}{2}(\bar{q}, \underline{C}\bar{q}) = \tfrac{1}{2}(\underline{S}\bar{q}', \underline{C}\,\underline{S}\bar{q}')$$
$$= \tfrac{1}{2}(\bar{q}', \underline{S}^{+}\underline{C}\,\underline{S}\,\bar{q}') \equiv \tfrac{1}{2}(\bar{q}', \underline{C}'\bar{q}')$$

where

$$\underline{C}' = \underline{S}^{-1}\underline{C}\,S$$

becomes the proper transformation of the P.E. matrix.

The equations of motion are still satisfied by this transformation,

$$\ddot{\bar{q}} = -\underline{C}\,\bar{q} \quad \rightarrow \quad \underline{S}\,\ddot{\bar{q}}' = -\underline{C}\,\underline{S}\,\bar{q}'$$

and

$$\ddot{\bar{q}}' = \underline{C}'\,\bar{q}'$$

The eigenvalues of $\underline{C}'$ are the same as $\underline{C}$ because our transformation has not change the physics of the problem.

PROBLEMS

1. Show eigenvalues of $\underline{U}$ have absolute value 1
where $\underline{U}' = \underline{S}^{-1} U$

2. Show in a $\underline{U}$ ~~eigenva~~ equivalence transformation
a Hermitean matrix stays Hermitean.

3. Assume eigenvectors U^n and eigenvalues λ^n of $\underline{A}$ (hermetean)
are known. Solve for $\bar{x}$ in terms of $\bar{y}$ the equation

$$\underline{A}\,\bar{x} - \lambda\,\bar{x} = \bar{y}$$

where λ = number.
Suggest a physical problem

4. The eigenvalues eq. for eigenvalues of $\underline{A}$ is
$\det(\underline{A} - \lambda \underline{I}) = 0$ is a polynominal
equation for λ.

$$P(\lambda) = \Sigma\, C_n \lambda^n = 0$$

Show matrix $\underline{A}$ satisfies

$$\sum_n C_n \underline{A}^n = 0$$