Hamitonian's principle

$$I[q(t)] = \int_{t_0}^{t_0} \mathcal{L}(q, \dot{q}, t)\, dt$$

(1) If $I[q(t)]$ is stationary then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

(2) If $\tilde{\mathcal{L}} = L + \frac{d}{dt} F(q,t)$, then $\frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}}\right) - \frac{\partial \tilde{\mathcal{L}}}{\partial q} = 0$

Symmetry $\quad = \mathcal{L}(q, \dot{q}, t) + \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial t}$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial F}{\partial q} \qquad \cancel{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right.} + \left.\frac{\partial F}{\partial q}\right) - \cancel{\frac{\partial \mathcal{L}}{\partial q}} - \frac{\partial}{\partial q}\left(\frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial t}\right) = 0$$

$$\cancel{\frac{\partial F}{\partial q \partial q} \cdot \dot{q}} + \cancel{\frac{\partial}{\partial q}\frac{\partial F}{\partial t}} - \cancel{\frac{\partial}{\partial q}\frac{\partial F}{\partial q} \cdot \dot{q}} - \cancel{\frac{\partial}{\partial q}\frac{\partial F}{\partial t}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \underline{q}} \equiv \nabla_{\underline{q}} \mathcal{L} \quad : \text{ index notation}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \underline{q}}\right)_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} \equiv \nabla_{\dot{q}} \mathcal{L}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad \dot{q}_1 = \frac{\partial H}{\partial p_1} \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} \cdots \dot{q}_n = \frac{\partial H}{\partial p_n}$$

$$q = q(s,t)$$

$$s = s(q,t)$$

$$L(q,\dot{q},t) = L\left(q(s,t), \frac{d}{dt}q(s,t), t\right)$$

$$= \tilde{L}(s, \dot{s}, t)$$

$$\dot{q}_i = \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}$$

$$\frac{\partial \dot{q}_i}{\partial \dot{s}_i} = \frac{\partial q_i}{\partial s_i}$$

We want to show that

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{s}} - \frac{\partial \tilde{L}}{\partial s} = 0$$

$$\frac{\partial \tilde{L}}{\partial \dot{s}^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \dot{s}^i} = \frac{\partial L}{\partial \dot{q}^j}\cdot\frac{\partial q^j}{\partial s^i}$$

$$\frac{\partial \tilde{L}}{\partial s^j} = \frac{\partial L}{\partial q^j}\frac{\partial q^j}{\partial s^i} + \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial s^i}$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{s}^i}\right) - \frac{\partial \tilde{L}}{\partial s^i} = \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^j}\cdot\frac{\partial q^j}{\partial s^i}\right) - \frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial s^i} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial s^i}$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^i}\right)\frac{\partial q^j}{\partial s^i}$$